Course NMA - Numerical Methods in Acoustics

Session 2 – Time domain methods

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About Lecturer: Bio

2004	**M.Sc.** Building Physics Eindhoven University of Technology
2007	University of Mississippi National Center for Physical Acoustics
2009	**Ph.D.** Applied Acoustics Chalmers University of Technology, Gothenburg
2009-2011	**Post-Doc** Aero-Acoustics KU Leuven
2011-2013	**Senior Researcher** Applied Acoustics Chalmers University of Technology, Gothenburg
2012- now	**Assistant professor** Acoustics of the Built Environment Eindhoven University of Technology

About Lecturer: Research interests

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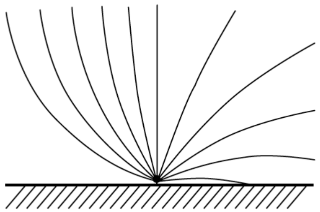
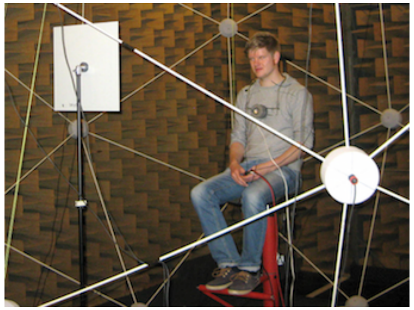
- Development of numerical methods for the Built Environment
  - Parabolic Equation [1]
  - Equivalent Sources Method (ESM) [2]
  - Pseudospectral time-domain method (PSTD) [3]
  - Discontinuous Galerkin method (DG) [4]

- Application of numerical methods in the Built Environment

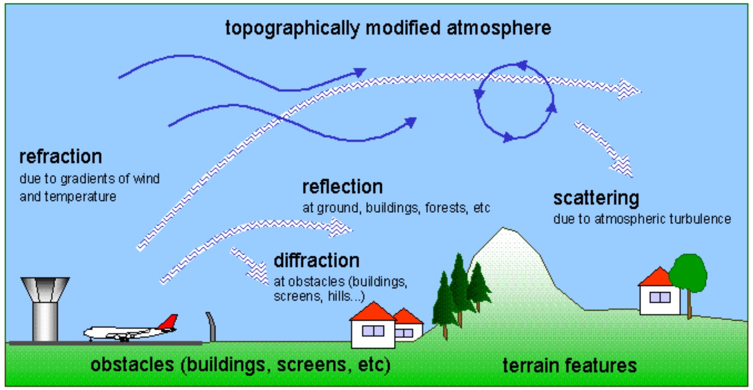


- Human echolocation



Scope of Lecture

- **Propagation of sound in fluids,** areas of application are acoustics of indoor and outdoor spaces (room acoustics and outdoor sound propagation)
    - Other areas of application of time-domain methods are amongst others propagation in porous media, underwater acoustics, propagation in solids (structural vibrations, seismic wave propagation)



http://www.pa.op.dlr.de/acoustics/index.html

Scope of Lecture

- **Time-domain wave-based acoustics methods**, i.e. direct numerical solutions of governing acoustics equations
- Role of **wave-based** acoustics methods in applications
  - Indoors: low frequency range, wave effects (interferences, seat-dip)
  - Outdoors: low frequency range (urban areas), complex meteorological effects
  - Auralisation
- **Time-domain** methods versus frequency domain methods
  - Broadband problems -> time domain is in favour
  - Tonal components dominate/modal analysis

    -> frequency domain is in favour
  - Auralisation -> time domain is in favour (inverse Fourier transform from frequency domain results is possible as well)
  - Non-linear problems -> time domain is in favour (frequency domain is typically solving time-harmonic acoustic equations)

Session 2

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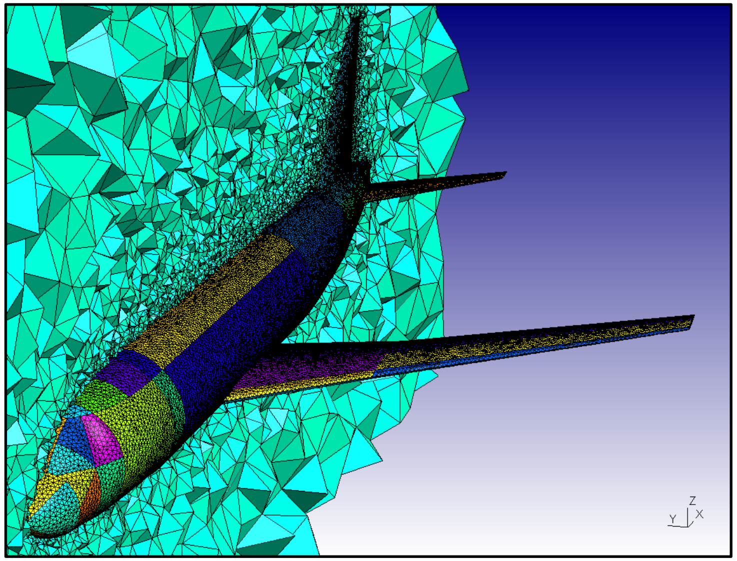
Course NMA

Scope of Lecture

- **Volume discretization methods,** i.e. discretization of the whole propagation medium
- Volume versus boundary discretization methods
  - Volume discretization favourable for non-homogeneous propagation media, domain discretization might be favourable for homogeneous and non-moving propagation media.
  - Volume discretization: sparse matrices

    Boundary discretization methods: dense matrices



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Contents

1. Governing equations

- Conservation equations, Navier-Stokes

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = \mathrm{Q}$$ mass

$$\frac{\partial\rho\boldsymbol{u}}{\partial t} + (\nabla\cdot\rho\boldsymbol{u})\boldsymbol{u} + \nabla p = \boldsymbol{f} + \nabla\vec{\boldsymbol{\tau}}$$ momentum

$$\frac{\partial\rho e}{\partial t} + \nabla\cdot(\rho e\boldsymbol{u}) = -\nabla(\boldsymbol{u}\mathrm{p}) + \nabla\cdot(\boldsymbol{u}\vec{\boldsymbol{\tau}}) + \nabla\cdot\boldsymbol{q} + f_e$$ energy

ρ	density
e	energy
$\boldsymbol{u}$	velocity
p	pressure
$\vec{\boldsymbol{\tau}}$	viscous stress tensor
Q	mass source term
$\boldsymbol{q}$	heat flux
$\boldsymbol{f}$	external body forces
f_e	external energy forces

- Why not solving for all physical processes together in outdoor acoustics?

1. Governing equations

- Ideal gas law

$$e(p,\rho,\boldsymbol{u})$$
$$\mathrm{T}(p,\rho)$$

- Viscosity may be neglected -> Euler equations

$$\vec{\boldsymbol{\tau}} = 0$$

- Decomposing physical variables in Mean and Turbulent components + Acoustic fluctuation (primed variables)

$$p = p_0 + p'$$
$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}'$$
$$\rho = \rho_0 + \rho'$$

- Linearizing equations with respect to the acoustic variables, i.e. insert variables above in Navier-Stokes equations, and neglect non-linear acoustic terms, as e.g.

$$u_x{'}\frac{\partial u_x{'}}{\partial x} = 0$$

1. Governing equations

- Incompressibility of the flow

$$\nabla \cdot \boldsymbol{u}_0 = 0$$

- Isentropic assumption, with c the adiabatic speed of sound

$$c_0 \rho' = p'$$

- This leads to the following form of the **linearized Euler equations (LEE)**, used for outdoor sound propagation

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 (\nabla \cdot \boldsymbol{u}') + (\boldsymbol{u}_0 \cdot \nabla) p' = c_0^2 \mathrm{Q}$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + (\boldsymbol{u}' \cdot \nabla)\boldsymbol{u}_0 + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}' + \frac{1}{\rho_0} \nabla p' = \frac{\boldsymbol{f}}{\rho_0}$$

1. Governing equations

- Indoor acoustics, $\boldsymbol{u}_0 = 0$, coupled linear acoustic equations

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 (\nabla \cdot \boldsymbol{u}') = c_0^2 \mathrm{Q}$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \frac{1}{\rho_0} \nabla p' = \frac{\boldsymbol{f}}{\rho_0}$$

- Wave equation (WE)

$$\frac{\partial^2 p'}{\partial t^2} - c_0^2 \Delta p' = c_0^2 \frac{\partial \mathrm{Q}}{\partial t} - \nabla \left(\frac{\boldsymbol{f}}{\rho_0} \right)$$

- LEE or WE?
  - Modelling of multiple propagation media, e.g. porous materials -> LEE
  - Some numerical methods favour LEE
  - WE requires less storage capacity (less variables to solve)

2. Time-domain solution methods

- Coupled linear acoustic equations

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 (\nabla \cdot \boldsymbol{u}') = 0$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \frac{1}{\rho_0} \nabla p' = 0$$

- Numerical solution of coupled linear acoustic equations

$$\hat{p}' \approx p'$$
$$\hat{\boldsymbol{u}}' \approx \boldsymbol{u}'$$

- We will further use the notation p' and $\boldsymbol{u}'$ as variables in the numerical calculations

2. Time-domain solution methods

Four methods for computing the spatial derivatives

Method	Mesh	Derivatives calculation	Order of method	DOF per wavelength	Parallel implemen-tation
Finite difference (FD)	Structured	Compact stencil	Variable	10 (Low order) 4 (High order)	[5]
Finite Volume (FV)	(Non-) Structured	From two elements	Low	10	[6]
Pseudospectral (PS)	Structured	Global (Fourier) Per element (Chebyshev)	Fixed (Fourier) Variable (Chebyshev)	2 (Fourier) Down to π (Chebyshev)	[7]
Discontinuous Galerkin (DG)	(Non-) Structured	Per element	Variable (per element)	Down to π (large elements, high order)	[8]

2. Time-domain solution methods

- Semi-discrete equation

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 (\nabla \cdot \boldsymbol{u}') = 0 \qquad \Rightarrow \qquad \frac{\partial p'}{\partial t} + \rho_0 c_0^2 \left(\widehat{\nabla \cdot \boldsymbol{u}'}\right) = 0$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \frac{1}{\rho_0} \nabla p' = 0 \qquad \Rightarrow \qquad \frac{\partial \boldsymbol{u}'}{\partial t} + \frac{1}{\rho_0} \widehat{\nabla p}' = 0$$

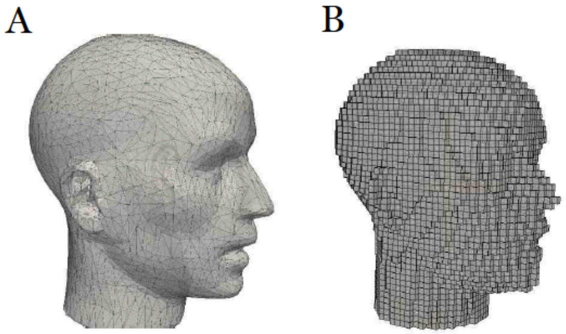
- with

$$\widehat{\nabla \cdot \boldsymbol{u}'} \approx \nabla \cdot \boldsymbol{u}'$$

$$\widehat{\nabla p}' \approx \nabla p'$$

2. Time-domain solution methods, FD

- Structured grid



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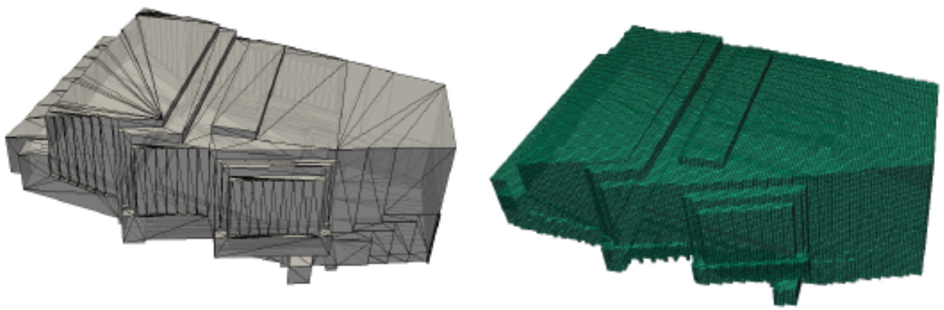

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2. Time-domain solution methods, FD

- Coupled linear acoustic semi-discrete equations

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \left(\widehat{\nabla \cdot \boldsymbol{u}'}\right) = 0$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \frac{1}{\rho_0} \widehat{\nabla p}' = 0$$

- 2D equations

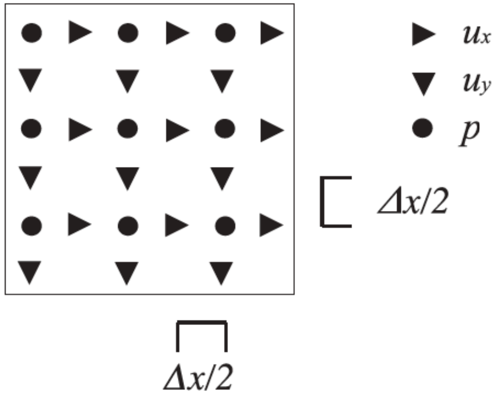
$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \left(\widehat{\frac{\partial u_x{}'}{\partial x}} + \widehat{\frac{\partial u_z{}'}{\partial z}}\right) = 0$$

$$\frac{\partial u_x{}'}{\partial t} + \frac{1}{\rho_0} \widehat{\frac{\partial p'}{\partial x}} = 0$$

$$\frac{\partial u_z{}'}{\partial t} + \frac{1}{\rho_0} \widehat{\frac{\partial p'}{\partial z}} = 0$$

2. Time-domain solution methods, FD

- Second order accurate scheme, staggered stencil



- 2D equations [11]

$$\left.\widehat{\frac{\partial u'_x}{\partial x}}\right|_{i,j} = \frac{u'_{x,i+1/2,j} - u'_{x,i-1/2,j}}{\Delta x}$$

$$\left.\widehat{\frac{\partial p'}{\partial x}}\right|_{i+1/2,j} = \frac{p'_{i+1,j} - p'_{i,j}}{\Delta x}$$

With spatial discretization
$(x, z) = (i\Delta x, j\Delta z)$

2. Time-domain solution methods, FD

- **Higher order** schemes with help of Taylor series

$$u'_{x,i+1/2,j} = u'_{x,i,j} - \frac{\Delta x}{2}\frac{\partial u'_{x,i,j}}{\partial x} + \frac{\Delta x^2}{8}\frac{\partial^2 u'_{x,i,j}}{\partial x^2} - \frac{\Delta x^3}{48}\frac{\partial^3 u'_{x,i,j}}{\partial x^3} + \cdots$$

- Also for $u'_{x,i-1/2,j}$, $u'_{x,i+3/2,j}$, $u'_{x,i-3/2,j}$

- Leads to the fourth order accurate derivative approximation, i.e. the error is in the order of $\Delta x^4 \frac{\partial^5 u'_x}{\partial x^5}$

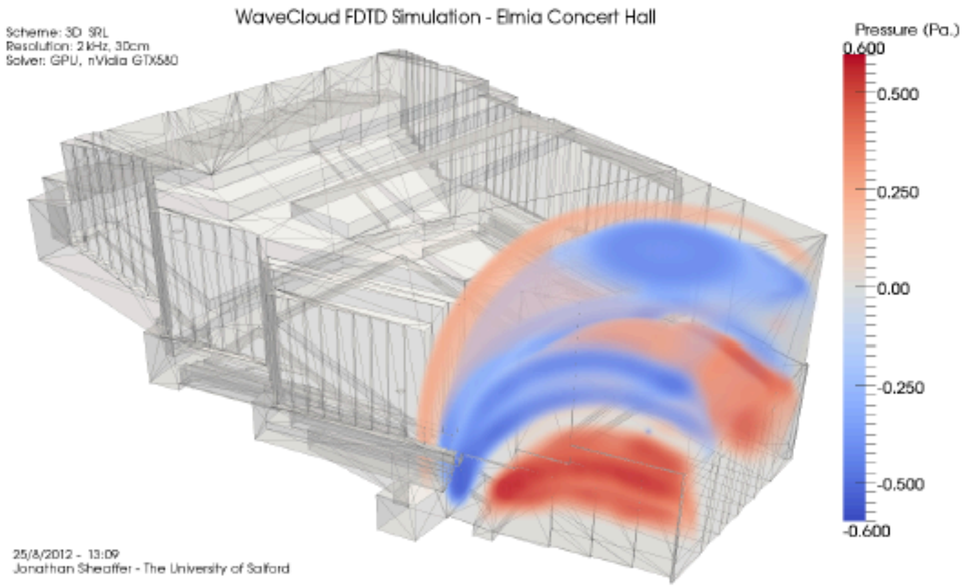
$$\left.\widehat{\frac{\partial u'_x}{\partial x}}\right|_{i,j} = \frac{-u'_{x,i+3/2,j} + 27u'_{x,i+1/2,j} - 27u'_{x,i-1/2,j} + u'_{x,i-3/2,j}}{24\Delta x}$$

- It means, that for a higher order scheme (higher accuracy), the stencil gets larger

2. Time-domain solution methods, FD

Consequences of FD method

- Easy to implement
- Order of accuracy can be changed
- Parallelizable
- Drawback of structured grid



[10]

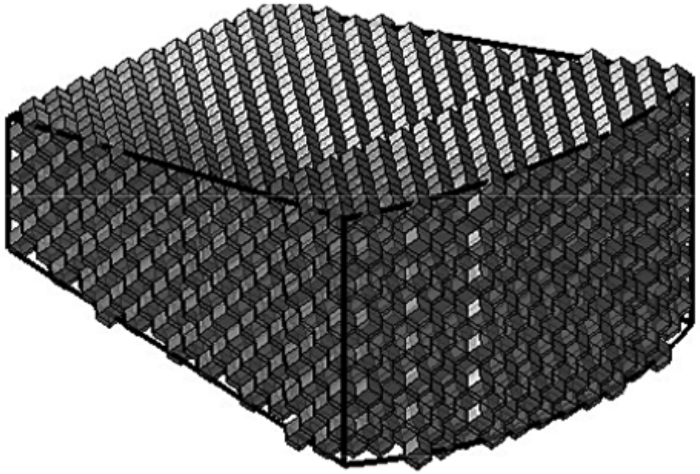
2. Time-domain solution methods, FD

Developments

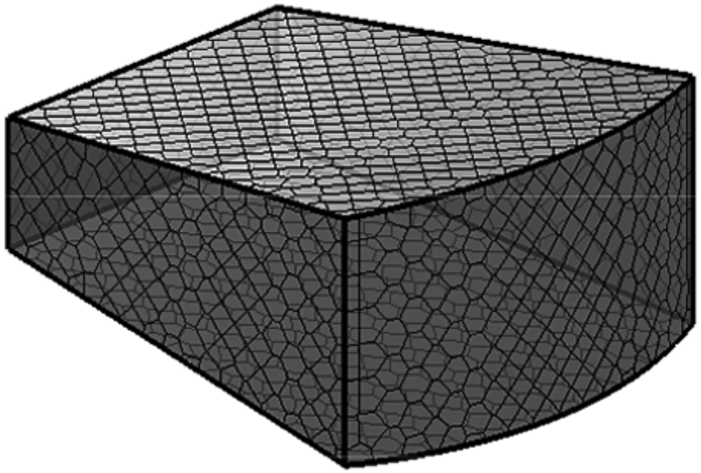
- **Compact implicit schemes**, where coefficients in the finite-difference scheme are matched to fulfil a certain order of accuracy [12]
- **Optimized schemes**, where coefficients in the finite-difference scheme are modified to produce reduced errors over a certain range of wave numbers. [13,14]
- Meteorological effects [15]
- Curvilinear implementation [16]
- Impedance boundary conditions [17]
- Parallel implementation [5]
- Open source software: http://www.ee.bgu.ac.il/~sheaffer/wavecloud.html

2. Time-domain solution methods, FV

- Structured or unstructured grid



Cubes [18]



Rhombic dodecahedral cells

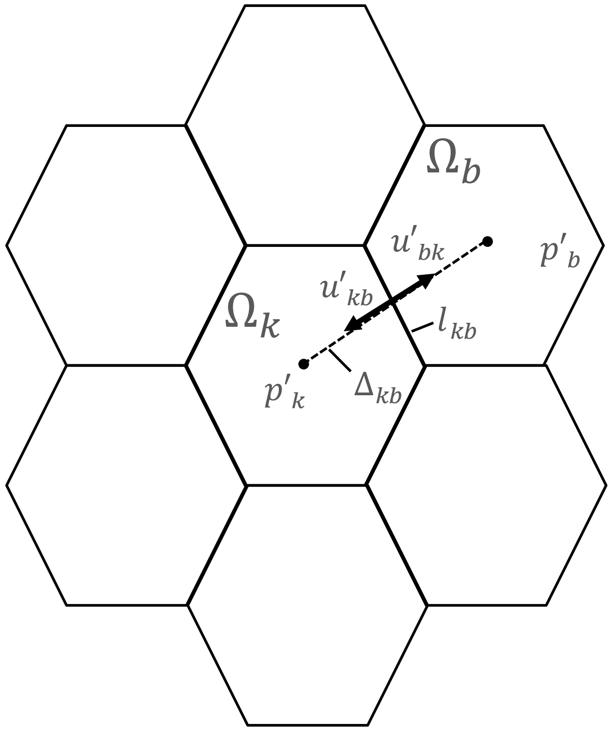
2. Time-domain solution methods, FV

- Coupled linear acoustic equations

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \left(\widehat{\nabla \cdot \boldsymbol{u}'}\right) = 0$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \frac{1}{\rho_0} \widehat{\nabla p'} = 0$$

- Volume integration per cell



2. Time-domain solution methods, FV

- Volume integration per cell in mass equation [19]

$$\int_{\Omega_k} \left(\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \left(\widehat{\nabla \cdot \boldsymbol{u}'} \right) \right) d\boldsymbol{x} \; = 0$$

- Making use of Gauss' theorem, i.e.

$$\int_{\delta\Omega_k} (\boldsymbol{u}' \cdot \boldsymbol{n}) dx \; = \int_{\Omega_k} \left(\widehat{\nabla \cdot \boldsymbol{u}'} \right) d\boldsymbol{x}$$

- Gives

$$\int_{\Omega_k} \left(\frac{\partial p'}{\partial t} \right) d\boldsymbol{x} + \rho_0 c_0^2 \int_{\delta\Omega_k} (\boldsymbol{u}' \cdot \boldsymbol{n}) dx = 0$$

2. Time-domain solution methods, FV

- For the hexagonal cell with 6 boundaries:

$$\int_{\Omega_k} \left(\frac{\partial p'}{\partial t}\right) d\boldsymbol{x} + \rho_0 c_0^2 \sum_{b=1}^{b=6} l_{kb} {u'}_{kb} = 0$$

$$S_k \frac{\partial {p'}_k}{\partial t} + \rho_0 c_0^2 \sum_{b=1}^{b=6} l_{kb} {u'}_{kb} = 0$$

With

${p'}_k$ = single pressure value if element k

S_k = the surface area of element *k*

${u'}_{kb} = -{u'}_{bk}$

- Momentum equation

$$\frac{\partial {u'}_{kb}}{\partial t} + \frac{1}{\rho_0} \frac{{p'}_b - {p'}_k}{\Delta_{kb}} = 0$$

2. Time-domain solution methods, FV

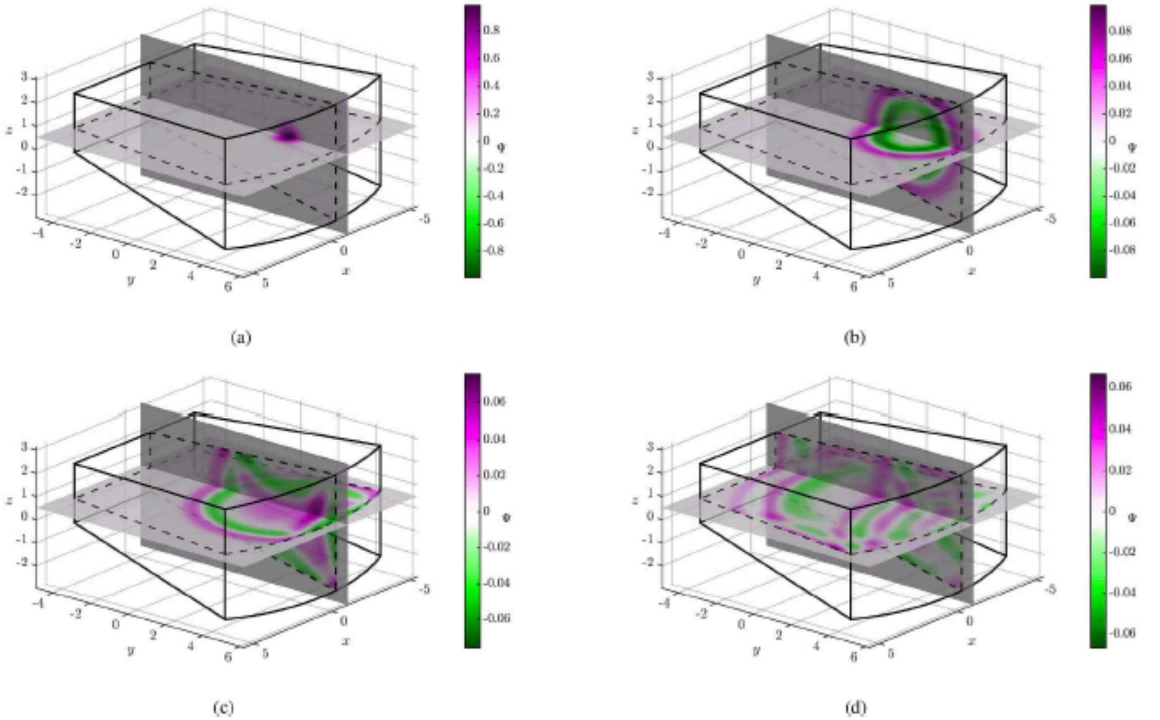
Consequences of FD method

- Unstructured grid possible
- Energy conservative (favourable for stability)
- Low order method in nature
- Parallel implementation possible

2. Time-domain solution methods, FV

Developments

- Impedance boundary conditions [18]
- Parallel implementation [6]
- Open source code http://www.clawpack.org

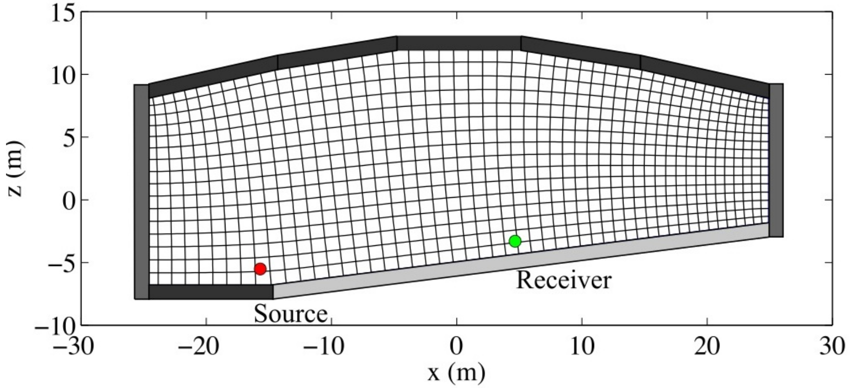


[18]

2. Time-domain solution methods, PS

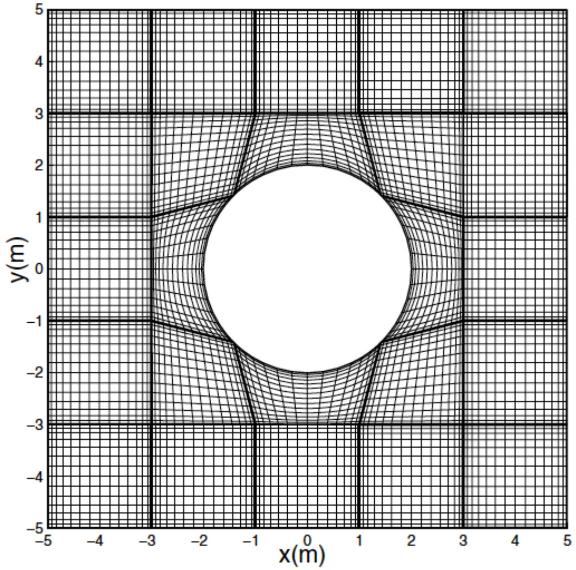
- Structured grid

Fourier PS



[20]

Chebyshev PS



[21]

2. Time-domain solution methods, FPS

- Coupled linear acoustic equations

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \left(\widehat{\nabla \cdot \boldsymbol{u}'}\right) = 0$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \frac{1}{\rho_0} \widehat{\nabla p'} = 0$$

- Compact 2D notation

$$\frac{\partial \boldsymbol{s}'}{\partial t} = jR_i^{-1} L \boldsymbol{s}'$$

$$\boldsymbol{s}' = [u_x', u_z', p']^T$$

$$L = \begin{bmatrix} 0 & 0 & j\frac{\partial}{\partial x} \\ 0 & 0 & j\frac{\partial}{\partial z} \\ j\frac{\partial}{\partial x} & j\frac{\partial}{\partial z} & 0 \end{bmatrix}$$

$$R_i = \begin{bmatrix} \rho_i & 0 & 0 \\ 0 & \rho_i & 0 \\ 0 & 0 & \frac{1}{\rho_i c_i^2} \end{bmatrix}$$

2. Time-domain solution methods, FPS

- Eigenvalue equation (corresponds to coupled linear acoustic equations for monochromatic waves)

$$[L - \boldsymbol{\varepsilon} R_i]\boldsymbol{\psi}_{\pm} = 0$$

$\boldsymbol{\psi}_{\pm} =$ eigenfunctions

$\boldsymbol{\varepsilon} =$ eigenvalues

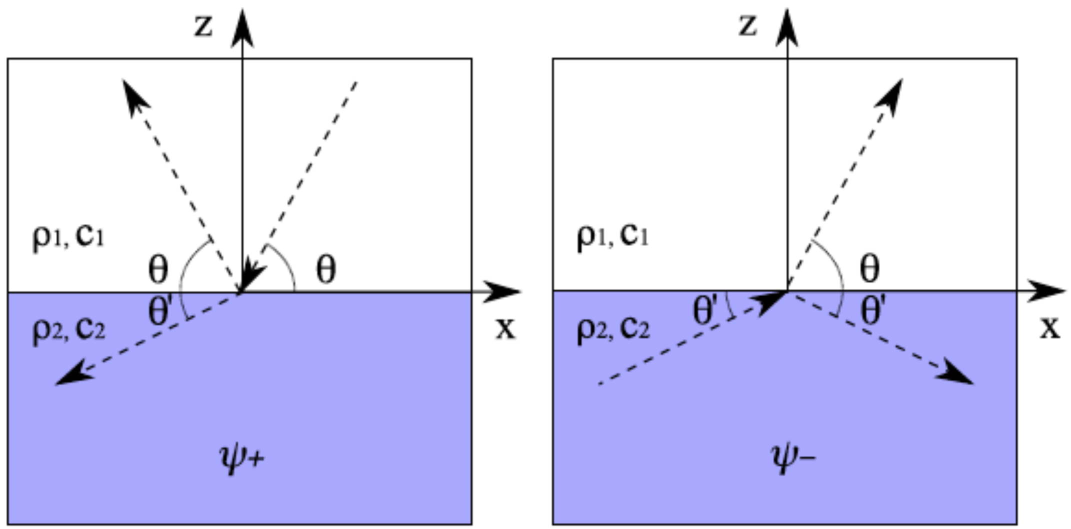
2. Time-domain solution methods, FPS

- Example of eigenfunctions for 2-media problem with interface at (x, 0)

$$[L - \boldsymbol{\varepsilon} R_i]\boldsymbol{\psi}_{\pm} = 0$$

$$\boldsymbol{\psi}_{\pm} = \text{eigenfunctions}$$

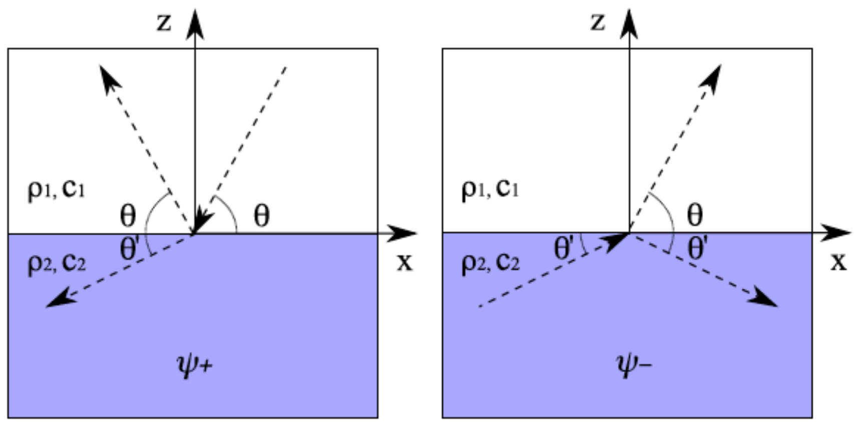
$$\boldsymbol{\varepsilon} = \quad \text{eigenvalues} = \varepsilon(-k_j c_j, k_j c_j)$$



2. Time-domain solution methods, FPS

- Example of eigenfunctions for 2-media problem with interface at (x, 0)

$$\boldsymbol{\psi}_{-}(\boldsymbol{kx}) = N_{-} = \begin{cases} \begin{bmatrix} \dfrac{k_x}{(k\rho c)_1}\left(\alpha_1 e^{-j\boldsymbol{k}_1\boldsymbol{x}} + \beta_1 e^{-j\overline{\boldsymbol{k}_1}\boldsymbol{x}}\right) \\ \dfrac{k_{z1}}{(k\rho c)_1}\left(\alpha_1 e^{-j\boldsymbol{k}_1\boldsymbol{x}} - \beta_1 e^{-j\overline{\boldsymbol{k}_1}\boldsymbol{x}}\right) \\ \alpha_1 e^{-j\boldsymbol{k}_1\boldsymbol{x}} + \beta_1 e^{-j\overline{\boldsymbol{k}_1}\boldsymbol{x}} \end{bmatrix}, & z \geq 0 \\ \begin{bmatrix} \dfrac{k_x}{(k\rho c)_2} e^{-j\boldsymbol{k}_2\boldsymbol{x}} \\ \dfrac{k_{z2}}{(k\rho c)_2} e^{-j\boldsymbol{k}_2\boldsymbol{x}} \\ e^{-j\boldsymbol{k}_2\boldsymbol{x}} \end{bmatrix}, & z \leq 0 \end{cases}$$



2. Time-domain solution methods, FPS

- Example of eigenfunctions for 2-media problem with interface at (x, 0)

$$\boldsymbol{\psi}_-(\boldsymbol{kx}) = N_- = \begin{cases} \begin{bmatrix} \dfrac{k_x}{(k\rho c)_1}\left(\alpha_1 e^{-j\boldsymbol{k}_1\boldsymbol{x}} + \beta_1 e^{-j\overline{\boldsymbol{k}_1}\boldsymbol{x}}\right) \\ \dfrac{k_{z1}}{(k\rho c)_1}\left(\alpha_1 e^{-j\boldsymbol{k}_1\boldsymbol{x}} - \beta_1 e^{-j\overline{\boldsymbol{k}_1}\boldsymbol{x}}\right) \\ \alpha_1 e^{-j\boldsymbol{k}_1\boldsymbol{x}} + \beta_1 e^{-j\overline{\boldsymbol{k}_1}\boldsymbol{x}} \end{bmatrix}, & z \geq 0 \\ \begin{bmatrix} \dfrac{k_x}{(k\rho c)_2} e^{-j\boldsymbol{k}_2\boldsymbol{x}} \\ \dfrac{k_{z2}}{(k\rho c)_2} e^{-j\boldsymbol{k}_2\boldsymbol{x}} \\ e^{-j\boldsymbol{k}_2\boldsymbol{x}} \end{bmatrix}, & z \leq 0 \end{cases}$$

- Coefficients α and β follow from physical interface conditions at z = 0

2. Time-domain solution methods, FPS

- Spatial operator $\boldsymbol{Ls'}$ calculated using the eigenfunctions

$$\boldsymbol{S'}_{\pm}(\boldsymbol{k},t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \boldsymbol{s}'(\boldsymbol{x},t)\,\overline{\boldsymbol{\psi}_{\pm}(\boldsymbol{k}_1,\boldsymbol{x})} R_i dx$$

$$L\boldsymbol{s}' = \sum_{\pm}\int_{-\infty}^{\infty}\int_{0}^{\infty} \epsilon R_i \boldsymbol{S'}_{\pm}(\boldsymbol{k},t)\,\boldsymbol{\psi}_{\pm}(\boldsymbol{k}_1,\boldsymbol{x})\,d\boldsymbol{k}$$

2. Time-domain solution methods, FPS

- Eigenfunctions are global: we only know them for simple cases
- For media with different densities, 1D Fourier transforms return
- Example for *dp'/dz* for the one-media case

$(\rho c)_1$

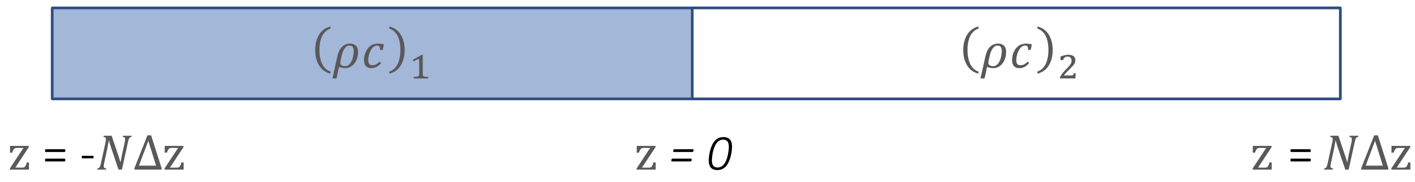
z = $-N\Delta z$ z = 0 z = $N\Delta z$

$$\left.\frac{\partial p'}{\partial z}\right|_{n\Delta z} = \mathcal{F}_z^{-1}\left(jk_z\mathcal{F}_z\ (p')\right) \qquad -N \leq n \leq N-1$$

- *N* the number of discrete points

2. Time-domain solution methods, FPS

- Example for *dp'/dz* for the two-media case



$$\left.\frac{\partial p'}{\partial z}\right|_{n\Delta z} = \mathcal{F}_z^{-1}\left(jk_z\mathcal{F}_z \begin{bmatrix} p'_1 \\ p'_2 \end{bmatrix}\right) \qquad \begin{matrix} -N \le n \le -1 \\ 0 \le n \le N-1 \end{matrix}$$

$$p'_1 = \begin{cases} p'_1(m\Delta z) & -N \le m \le -1 \\ R_{1,1}p'(-m\Delta z) + T_{2,1}p'(m\Delta z) & 0 \le m \le N-1 \end{cases}$$

$$p'_2 = \begin{cases} R_{2,2}p'(-m\Delta z) + T_{1,2}p'(m\Delta z) & -N \le m \le -1 \\ p'_2(m\Delta z) & 0 \le m \le N-1 \end{cases}$$

2. Time-domain solution methods, FPS

Consequences of PS method

- ‘High order’ method, low storage
- Acoustic variables should spatially be periodic
- Spectral accuracy up to $k = \pi/\Delta x$, i.e. down to $\lambda=2\Delta x$
- DFTs are operated by Fast Fourier Transforms (FFTs), with a computational overhead of log2(N) operations per discrete point for a 1D operation (with N the number of discrete points)

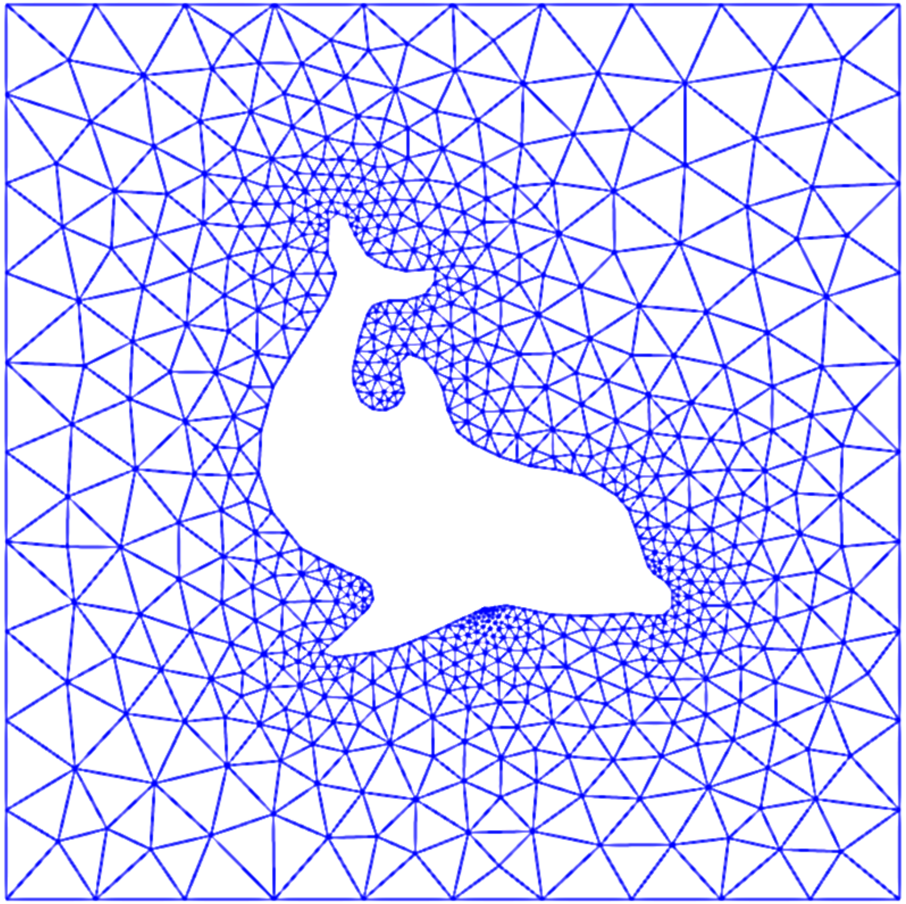
2. Time-domain solution methods, FPS

Developments

- Boundaries (material boundaries and non-reflecting boundaries, frequency dependent impedance conditions not possible yet) [3]
- Meteorological effects [3]
- Local grid refinement [22]
- Curvilinear implementation [23]
- Domain decomposition [24]
- Source directivity [25]
- Air absorption [26]
- GPU acceleration [7,24]
- Open source code www.openpstd.org

2. Time-domain solution methods, DG

- Discretization of the space by non-overlapping elements (unstructured grid)



http://hplgit.github.io/INF5620/doc/pub/fig-fem/dolfin_mesh.png

2. Time-domain solution methods, DG

- Coupled linear acoustic equations

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \left(\widehat{\nabla \cdot \boldsymbol{u}'}\right) = 0$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \frac{1}{\rho_0} \widehat{\nabla p}' = 0$$

- Conservative form, 2D

$$\frac{\partial \boldsymbol{s}'}{\partial t} + \frac{\partial \mathbf{A}_j \boldsymbol{s}'}{\partial x_j} = 0$$

$$\boldsymbol{s}' = [\rho_0 u_x', \rho_0 u_z', p']^T$$

$$\mathbf{A}_j = \begin{bmatrix} 0 & 0 & \delta_{x,j} \\ 0 & 0 & \delta_{z,j} \\ c_0^2 \delta_{x,j} & c_0^2 \delta_{z,j} & 0 \end{bmatrix}$$

2. Time-domain solution methods, DG

Session 2

Maarten Hornikx

Course NMA

- Decompose variable vector onto (Lagrange) polynomials l per element Ω

$$\boldsymbol{s}'^{\Omega}(\boldsymbol{x},t) = \sum_{i=1}^{N_p} \boldsymbol{s}'^{\Omega}(\boldsymbol{x}_i,t)\, l_i^{\Omega}(\boldsymbol{x}) = \sum_{i=1}^{N_p} \boldsymbol{s}'^{\Omega}_i\, l_i^{\Omega}$$

with

$$N_p = \prod_{i=1}^{D} (p+i)/i$$

p = order

D = dimensionality

2. Time-domain solution methods, DG

- Insert this in coupled linear equations, multiply by test function l and spatially integrate element-wise (FEM approach)

$$\int_{\Omega} l_h^k \sum_{i=1}^{N_p} \left(\frac{\partial \boldsymbol{s'}_i}{\partial t} l_i \right)^{\Omega} d\boldsymbol{x} + \int_{\Omega} l_h^k \sum_{i=1}^{N_p} \left(\frac{\partial A_j \boldsymbol{s'}_i l_i}{\partial x_j} \right)^{\Omega} d\boldsymbol{x} = 0$$

- Integrate second term by parts

$$\int_{\Omega} l_h^{\Omega} \sum_{i=1}^{N_p} \left(\frac{\partial \boldsymbol{s'}_i}{\partial t} l_i \right)^{\Omega} d\boldsymbol{x} - \int_{\Omega} \frac{\partial l_h^{\Omega}}{\partial x_j} \sum_{i=1}^{N_p} \left(A_j \boldsymbol{s'}_i l_i \right)^{\Omega} d\boldsymbol{x}$$

$$+ \int_{\delta\Omega} l_h^{\Omega} \sum_{i=1}^{N_p} \left(A_j \boldsymbol{s'}_i \right)^{\delta\Omega} l_i^{\Omega} \, d\delta\boldsymbol{x} = 0$$

2. Time-domain solution methods, DG

- Rewriting in matrix form

$$\frac{\partial \boldsymbol{s}'^{\Omega}}{\partial t} = \left(M_{ih}^{\Omega}\right)^{-1}\left(S_{r,ih}^{\Omega}\left(A_j \boldsymbol{s}'\right)^{\Omega} - \sum_{b=1}^{B} M_{ih}^{\delta\Omega,b} \boldsymbol{n} \cdot \left(A_j \boldsymbol{s}'\right)^{\Omega,b}\right)$$

$$with$$

$$M_{ih}^{\Omega} = \int_{\Omega} l_i^{\Omega} l_h^{\Omega}\, d\boldsymbol{x}$$

$$S_{r,ih}^{\Omega} = \int_{\Omega} l_h^{\Omega} \frac{\partial l_h^{\Omega}}{\partial x_j}\, d\boldsymbol{x}$$

$$M_{ih}^{\Omega} = \int_{\delta\Omega,b} l_i^{\Omega,b} l_h^{\Omega,b}\, d\delta\boldsymbol{x}$$

2. Time-domain solution methods, DG

- For elements with straight edges

$$M_{ih}^{\Omega} = |J^{\Omega}| M_{ih}^{\Delta} = |J^{\Omega}| \int_{\Delta} l_i^{\Delta} l_h^{\Delta} \, d\boldsymbol{x}$$

$$S_{r,ih}^{\Delta} = (J^{\Omega})_{rs}^{-1} |J^{\Omega}| \left(S_s^{\Delta}\right)_{ih} = (J^{\Omega})_{rs}^{-1} |J^{\Omega}| \int_{\Omega} l_h^{\Delta} \frac{\partial l_h^{\Delta}}{\partial \xi_s} \, d\boldsymbol{x}$$

$$M_{ih}^{\delta\Omega,b} = \left|J^{\delta\Omega,b}\right| M_{ih}^{\delta\Delta,b} = \left|J^{\delta\Omega,b}\right| \int_{\delta\Delta,b} l_i^{\Delta,b} l_h^{\Delta,b} \, d\delta\boldsymbol{x}$$

- With reference element Δ, having coordinates ξ_1, ξ_2

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2. Time-domain solution methods, DG

- Flux term

$$F^{\delta\Omega,b} = \left(A_j \boldsymbol{s}'\right)^{\delta\Omega,b}$$

- Determines exchange of information between neighbouring elements and needs to be updated with information from other elements
- The outgoing flux inside an element (related to waves leaving the element) can be computed with the information in the element
- The incoming flux (related to waves propagating into the element) should be copied from neighbouring elements. This is the outgoing flux of the neighbouring elements.
- Fluxes can be computed with Riemann flux [27]:

$$F_R^{\delta\Omega,b} = \frac{1}{2}\left[\left(F_R^{\delta\Omega,b} + F_R^{\delta\Omega',b}\right)n_b - \alpha\left(\boldsymbol{s}'^{\delta\Omega,b} + \boldsymbol{s}'^{\delta\Omega',b}\right)\right]$$

- Other, more accurate way is through method of characteristics [28]

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2. Time-domain solution methods, DG

Consequences of DG method

- Small matrices to compute per element (in contrast to FEM)
- Efficient for parallelization
- Order of accuracy can be changed locally (smaller elements, higher order)
- Grid flexibility

2. Time-domain solution methods, DG

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Some developments

- Curved elements [29]
- Aero-acoustic applications [30]
- Parallel implementations [8]

2. Time-domain solution methods, Time iteration

- Fully discrete solution

$$\frac{\partial p'}{\partial t} + \rho_0 c_0^2 \left(\widehat{\nabla \cdot \boldsymbol{u}'}\right) = 0 \qquad \widehat{\frac{\partial p'}{\partial t}} + \rho_0 c_0^2 \left(\widehat{\nabla \cdot \boldsymbol{u}'}\right) = 0$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \frac{1}{\rho_0} \widehat{\nabla p'} = 0 \qquad \widehat{\frac{\partial \boldsymbol{u}'}{\partial t}} + \frac{1}{\rho_0} \widehat{\nabla p'} = 0$$

- The evaluation of the time derivative can be considered separate from the evaluation of the spatial derivatives.
- Analytical iteration in time (k-space method [31]) versus numerical in time (most methods).

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2. Time-domain solution methods, Time iteration

- **Implicit time iteration schemes** make use of information of acoustic variables at former and future time steps. Examples:
  - Alternating-direction implicit (ADI) scheme [32]
  - Split-step scheme [33]

- **Explicit time iteration schemes** make use of information of variables at former time steps only. Examples:
  - Leap-frog method [11]
  - Runge-Kutta method [14]

- Implicit methods have a higher computational demand (matrix inversion) and are harder to implement than explicit schemes
- Implicit methods can typically handle larger discrete time steps than explicit schemes with regard to accuracy and stability

2. Time-domain solution methods, Time iteration

- Often used scheme: **Leap-frog** explicit method

  (used in combination with FD [15] and FV [19] methods)

- Variables pressure and velocities are staggered in time:

  p'^{τ}, with discrete time $t = \tau\Delta t$ and

  $\boldsymbol{u}'^{\tau+1/2}$, with discrete time $t = (\tau + 1/2)\Delta t$

- This leads to calculation of derivatives as follows

$$\left.\frac{\widehat{\partial p'}}{\partial t}\right|_{\tau+1/2} = \frac{p'^{\tau+1} - p'^{\tau}}{\Delta t}$$

$$\left.\frac{\widehat{\partial \boldsymbol{u}'}}{\partial t}\right|_{\tau} = \frac{\boldsymbol{u}'^{\tau+1/2} - \boldsymbol{u}'^{\tau-1/2}}{\Delta t}$$

2. Time-domain solution methods, Time iteration

- **Leap-frog** method, second order accurate

First

$$\boldsymbol{u}'^{\tau+1/2} = \boldsymbol{u}'^{\tau-1/2} - \Delta t \frac{1}{\rho_0} \left(\widehat{\nabla p'}\right)^{\tau}$$

Then

$$p'^{\tau+1} = p'^{\tau} - \Delta t \rho_0 c_0^2 \left(\widehat{\nabla \cdot \boldsymbol{u}'}\right)^{\tau+\frac{1}{2}}$$

Again, the velocities can be updated

$$\boldsymbol{u}'^{\tau+3/2} = \boldsymbol{u}'^{\tau+1/2} - \Delta t \frac{1}{\rho_0} \left(\widehat{\nabla p'}\right)^{\tau+1}$$

And so on

2. Time-domain solution methods, Time iteration

- Often used scheme: **Runge-Kutta** (RK)method

  (used in combination with PS [3] and DG [27] methods)

- The RK method can be seen as a combination of first order derivative calculations at intermediate positions between two time steps Δt
- Compared to forward differences method, a) the order of the method increases, i.e. the error is reduced, and, b) stability is promoted
- RK method with p stages, with γ_i coefficients between 0 and 1

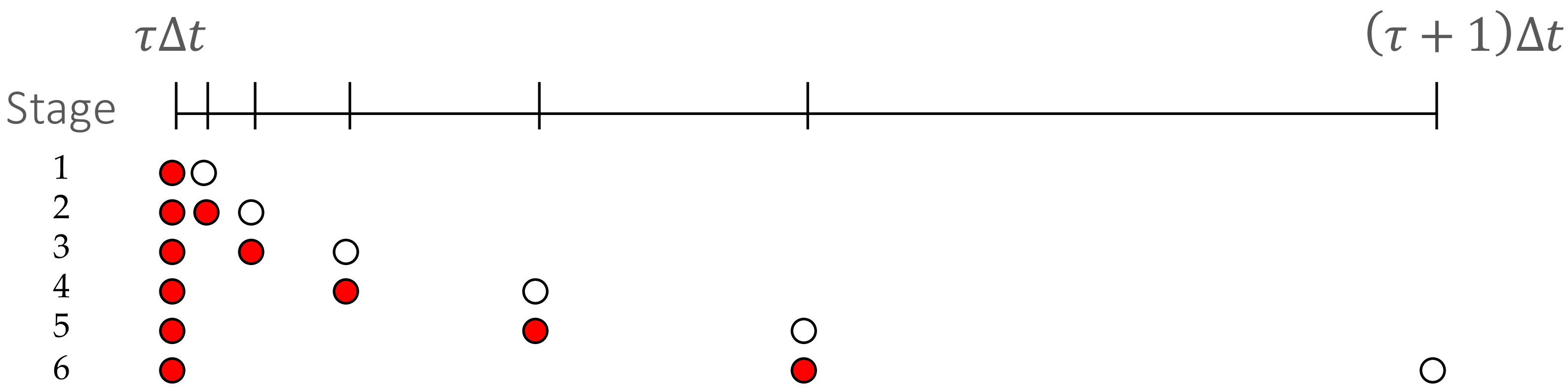
$$\boldsymbol{s}'^{\tau_0} = \boldsymbol{s}'^{\tau}$$
$$\boldsymbol{s}'^{\tau_i} = \boldsymbol{s}'^{\tau_0} - \gamma_i \Delta t \left(R_i^{-1} L \boldsymbol{s}'^{\tau_{i-1}} \right)$$
$$for \quad i = 1 \ldots p$$
$$\boldsymbol{s}'^{\tau+\Delta\tau} = \boldsymbol{s}'^{\tau_p}$$

- When the coefficients γ_i are found by matching Taylor series for $\boldsymbol{s}'^{\tau+\Delta\tau}$, the method has order p

2. Time-domain solution methods, Time iteration

- Often used scheme: **Runge-Kutta** (RK) method

  Example of low storage RK method (6 stages): only 2 former stage needs to be stored



Result stored at former time stage

New result

3. Boundary conditions

- Impedance of a propagation medium (may be a boundary medium)

$$Z_m = \frac{p'}{u'_n}$$

With
u'_n = acoustic velocity in direction of sound propagation

- At physical boundaries, impedance conditions should be specified in a calculation method. As the pressure and velocity component normal to the boundary are continuous, the surface impedance Z_s can be defined as

$$Z_s = \frac{p'}{u'_s} = Z_m$$

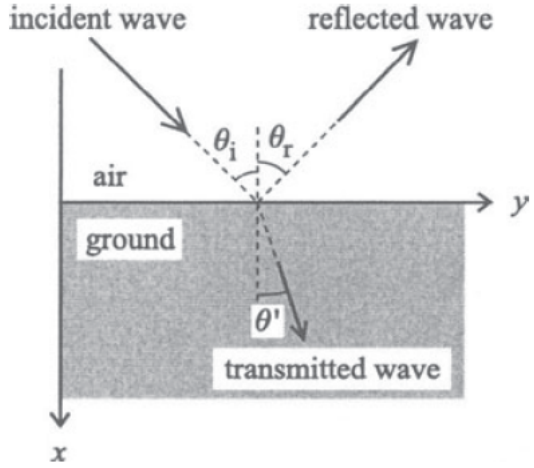
With
u'_s = acoustic velocity normal to the surface

- This is a locally reaction approach!

3. Boundary conditions

- Consequence of locally reacting approximation, plane wave reflection coefficient is obtained

$$R_{p,loc} = \frac{Z_s \cos(\theta_i) - Z_{air}}{Z_s \cos(\theta_i) + Z_{air}}$$



[34]

- Instead of the correct coefficient

$$R_p = \frac{\frac{Z_s}{\cos(\theta')} \cos(\theta_i) - Z_{air}}{\frac{Z_s}{\cos(\theta')} \cos(\theta_i) + Z_{air}}$$

3. Boundary conditions

- In numerical calculations, $Z_m = \frac{p'}{u'_n}$ can be imposed by either relating the acoustic pressure to the acoustic velocity at the boundary, or vice versa [11]
- Three conditions for frequency–domain impedance boudanry conditions to be valid for time-domain calculations [35]

1) real-valued acoustic variables in time domain

$$\overline{Z_s}(f) = Z_s(-f)$$

2) boundary absorbs acoustic energy and does not produce it

$$Re[Z_s(f)] \geq 0 \qquad \text{for } f > 0$$

3) The acoustic interaction with the boundary is causal

a) $Z_s(f) \geq 0$ is analytic in $Im[f] > 0$
b) $|Z_s(f)|$ is square integrable over the real f-axis
c) There is a real t_0 such that $Z_s(f)e^{-j\omega t_0} \to 0$ uniformly with regard to $Arg(f)$ for $|f| \to \infty$ in $Im[f] \geq 0$

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3. Boundary conditions

- Impedance condition in time domain

$$p'(t) = \int_0^t z_s(t-t')\, u'_s(t')dt'$$

With

$$z_s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_s(\omega)e^{j\omega t}d\omega$$

- This means that the velocity over previous time steps must be known!
- Efficient method is recursive convolution method [17]

$$Z_s(\omega) = \sum_{k=1}^{K}\frac{A_k}{\lambda_k - j\omega}$$

With

λ_k poles

K number of poles

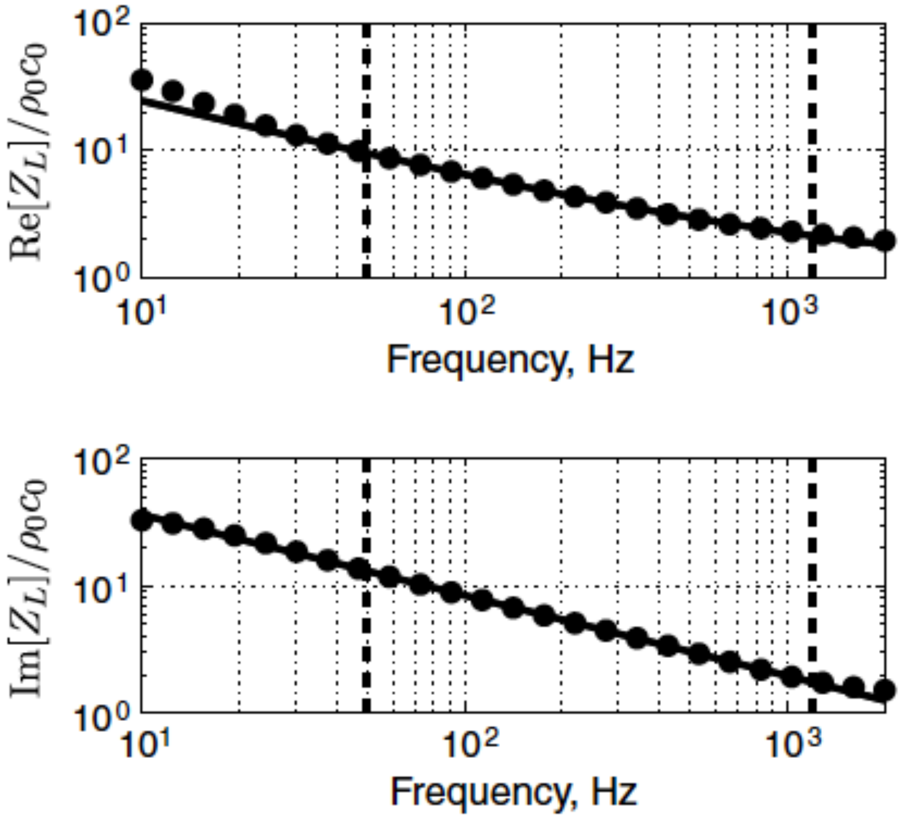
$$p'^{\tau} = \sum_{k=1}^{K} A_k \phi_k^{\tau}$$

with

$$\phi_k^{\tau} = u'^{\tau}_s \frac{1-e^{-\lambda_k \Delta t}}{\lambda_k} + \phi_k^{\tau-1}e^{-\lambda_k \Delta t}$$

3. Boundary conditions

- Normalized surface impedance from Miki impedance model and recursive method with $K = 5$ [17]. Line: Miki model, Dots: recursive method



3. Boundary conditions

- **Extended reacting boundaries**, modelling of propagation in boundary material e.g. for ground surface [34,36]
- Zwikker and Kosten equation for equivalent fluid representation of ground medium

$$\frac{\partial p'}{\partial t} + \frac{\rho_0 c_0^2}{\Omega}(\nabla \cdot \boldsymbol{u}') = 0$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \frac{\Omega}{c_s \rho_0}\nabla p' + \sigma \frac{\Omega}{c_s \rho_0}\boldsymbol{u}' = 0$$

With

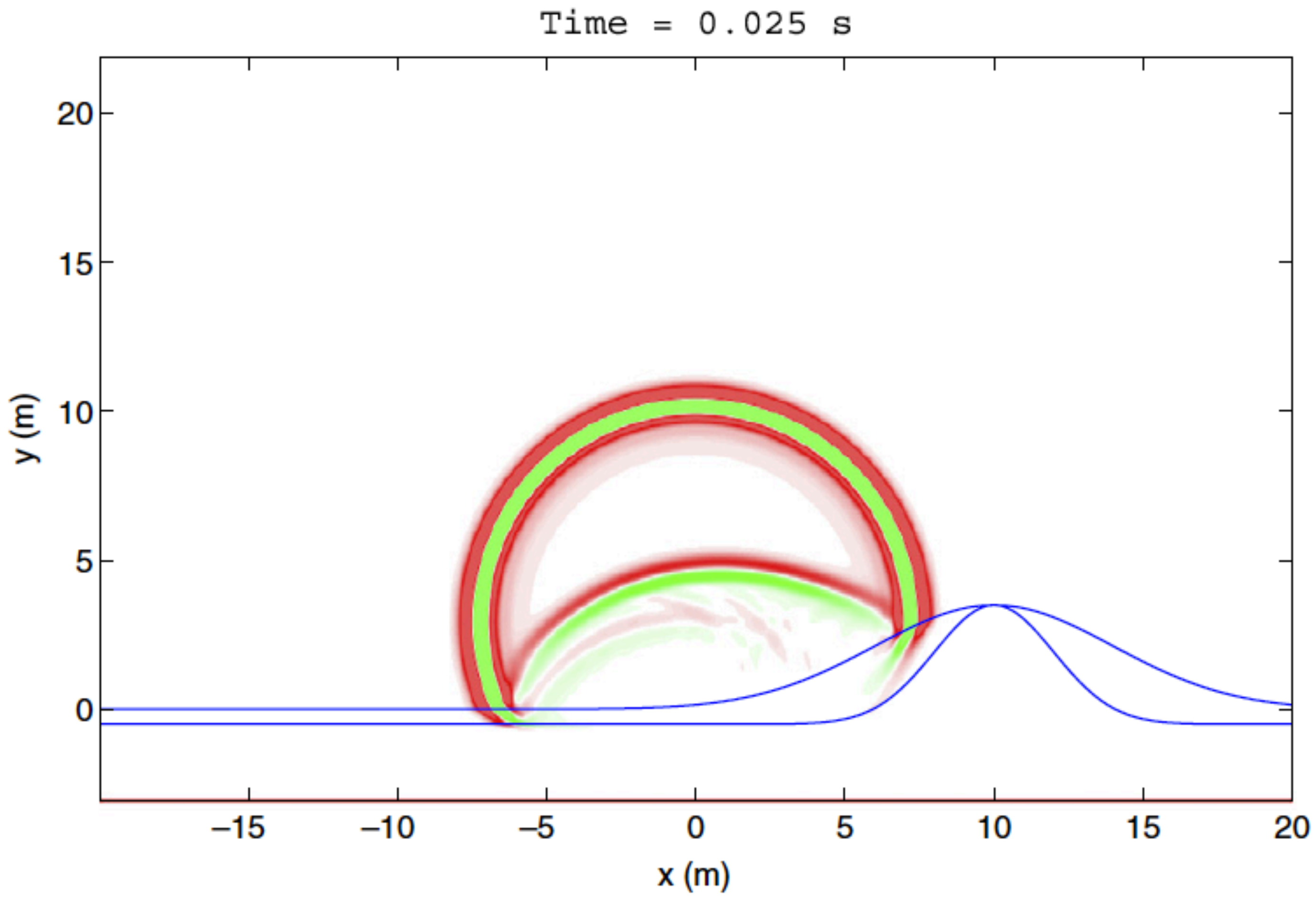
σ = flow resistivity

Ω = porosity

c_s = structure constant

3. Boundary conditions

- Equivalent fluid representation of ground medium [36]



3. Boundary conditions

- Perfecty matched layers (PML) for totally absorbing boundaries (as in outdoor propagation), i.e. [3]

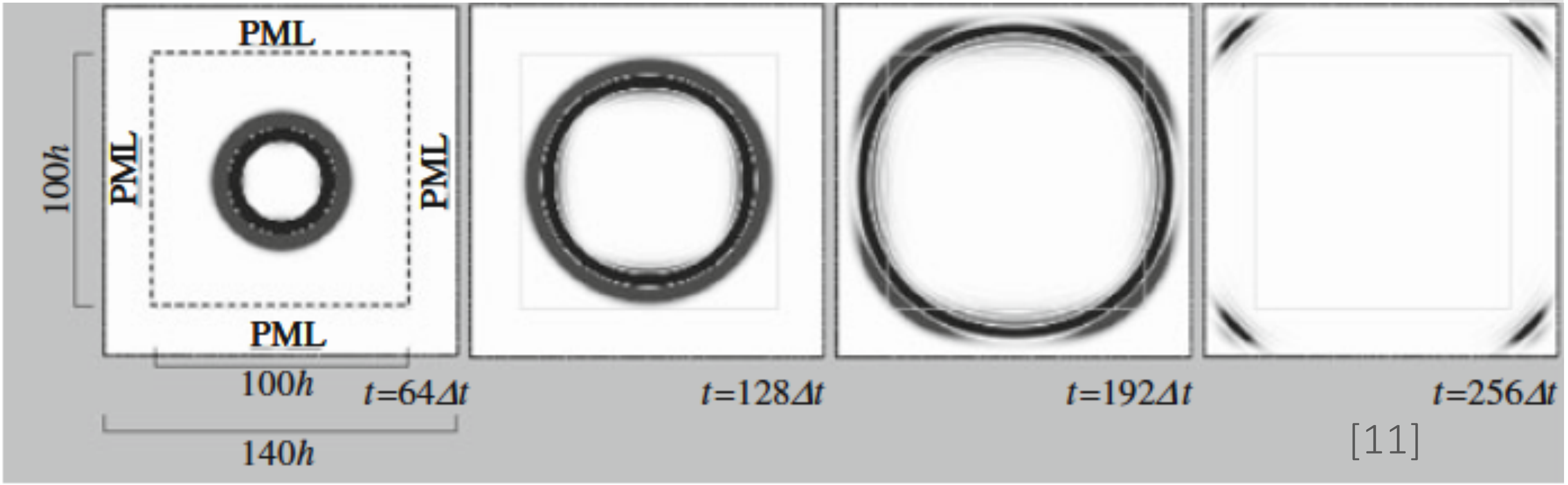
$$\frac{\partial p'_j}{\partial t} + \rho_0 c_0^2 \frac{\partial u'_j}{\partial x_j} + \sigma_{j,p} p'_j = 0$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \frac{1}{\rho_0} \nabla p' + \boldsymbol{\sigma}_u \boldsymbol{u}' = 0$$

With

$p' = p'_x + p'_z$

σ = PML coefficients



[11]

4. Errors and stability

- Errors in calculation (compared to measurement or reference calculation) may arise due to:
- Physical modelling
  - Approximation of geometry, e.g. Staircase approach of curved boundary, mesh too coarse to resolve small geometrical effects
  - Approximation of boundary conditions (impedances)
  - Erroneous conditions of propagation medium (temperature, wind field outdoors)
  - Source and receivers: deviation in location, deviation in directivity
- Numerical modelling
  - Error in spatial derivative operator
  - Error in time derivative operator
  - Signal analysis approximations, e.g. truncation of time signals

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4. Errors and stability

- Two types of errors, relative to a reference calculation/measurement

  Dissipation error is related to the accuracy in the energy of the acoustic variables

$$\epsilon_{amp}(f) = 20log_{10}\left(\frac{|\boldsymbol{S}'(f)| - |\boldsymbol{S}'_{ref}(f)|}{|\boldsymbol{S}'_{ref}(f)|}\right)$$

$$\boldsymbol{S}' = \int_0^T \boldsymbol{s}' e^{j\omega t} dt$$

  Dispersion error is related to the accuracy in the phase of the acoustic variables

$$\epsilon_{\phi}(f) = 20log_{10}\left(\frac{|\phi[\boldsymbol{S}'(f)] - \phi[\boldsymbol{S}'_{ref}(f)]|}{\pi}\right)$$

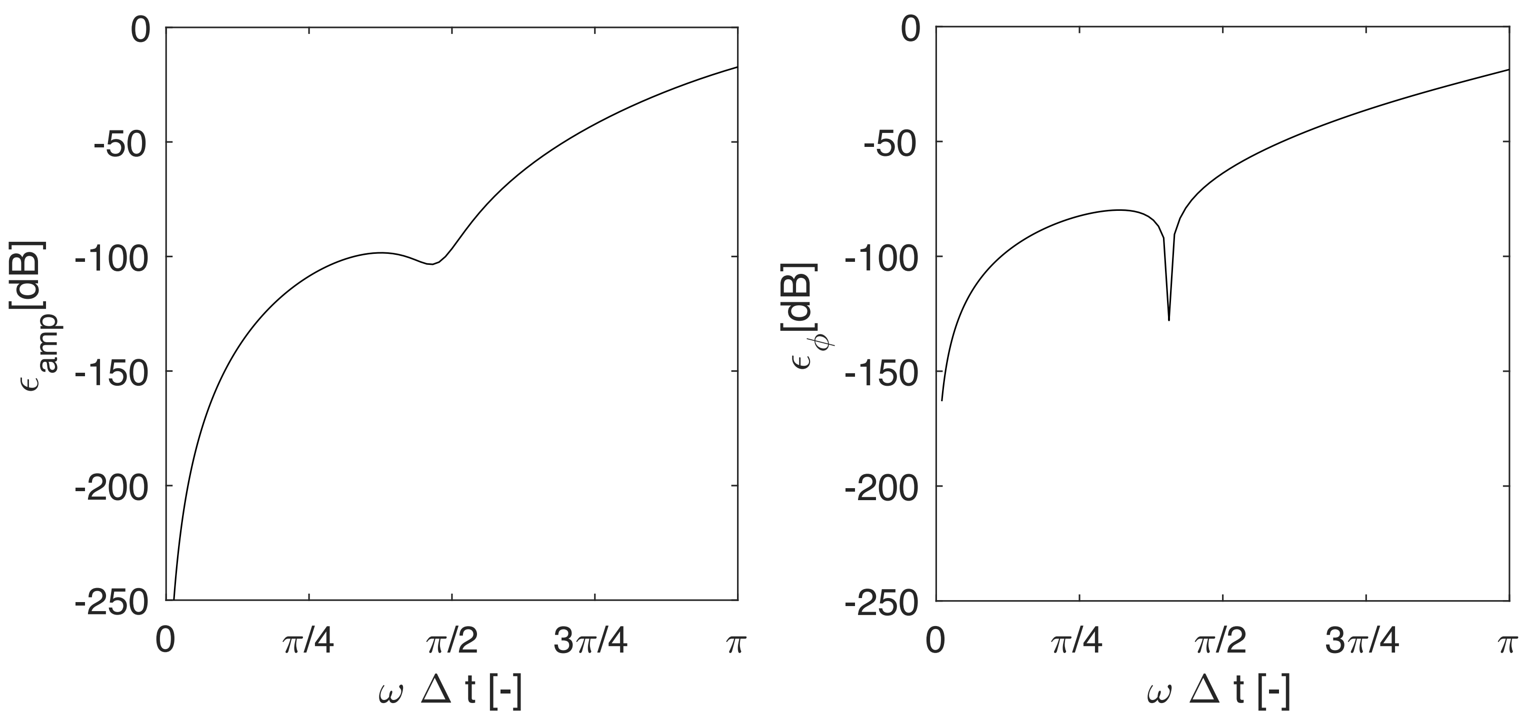
  $\phi[\boldsymbol{S}'(f)]$ = phase angle of $\boldsymbol{S}'(f)$ in radians

  Both errors are a function of frequency

4. Errors and stability

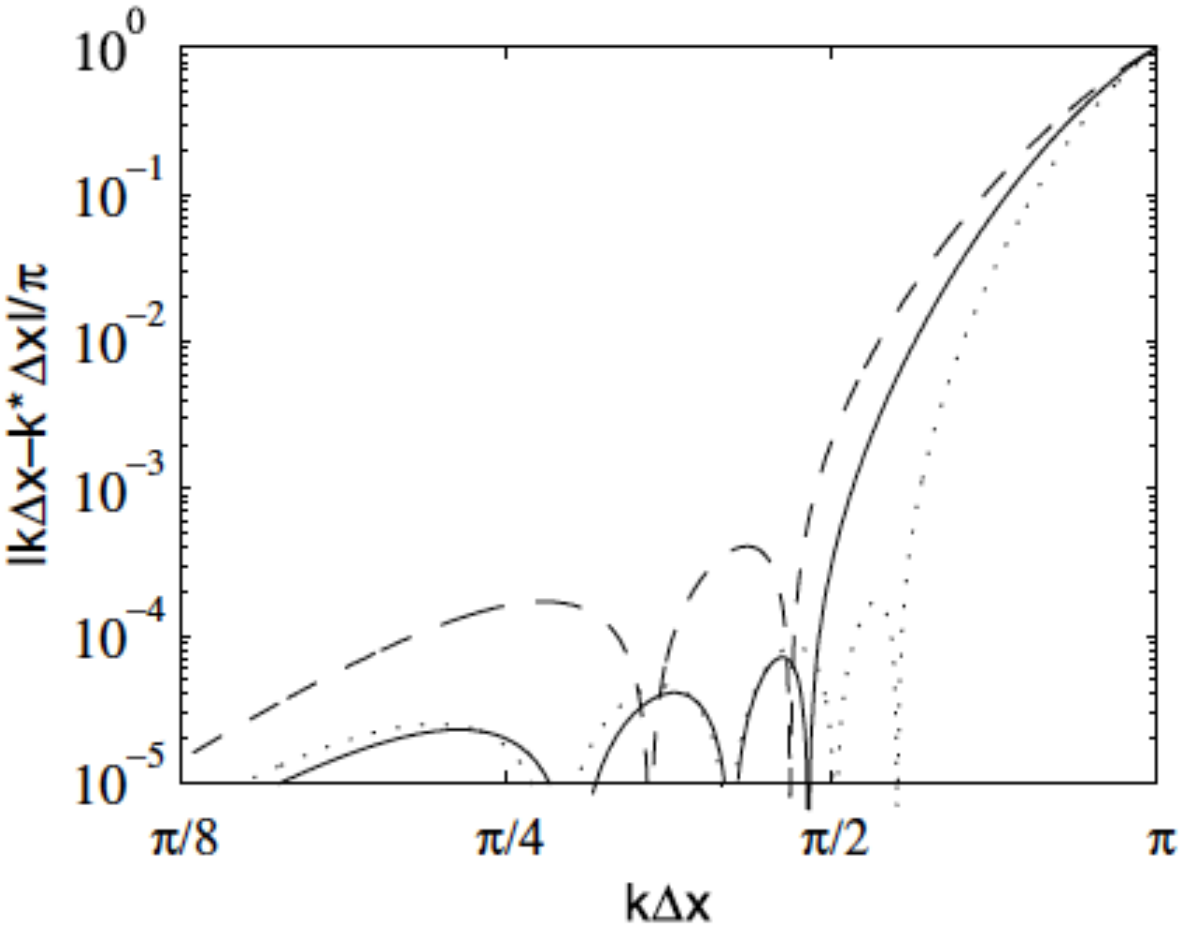
- Dissipation and dispersion error can be evaluated from a calculation with the full numerical method, but also from the spatial derivative operator or time derivative operator separately

Error of low storage optimized six-stage RK scheme, per time step



4. Errors and stability

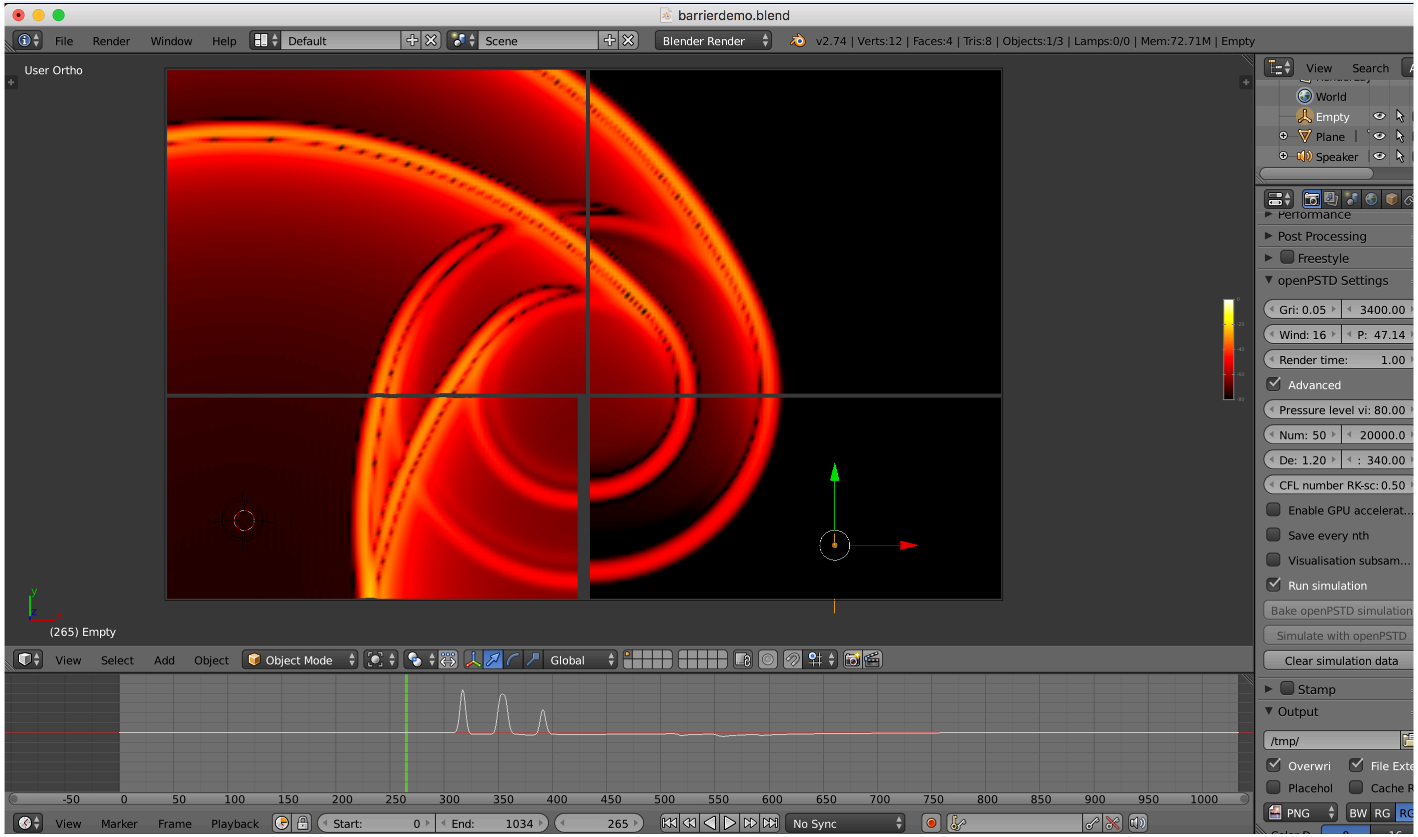
- Phase error of optimized FD schemes [14]



4. Errors and stability

- **Numerical stability**

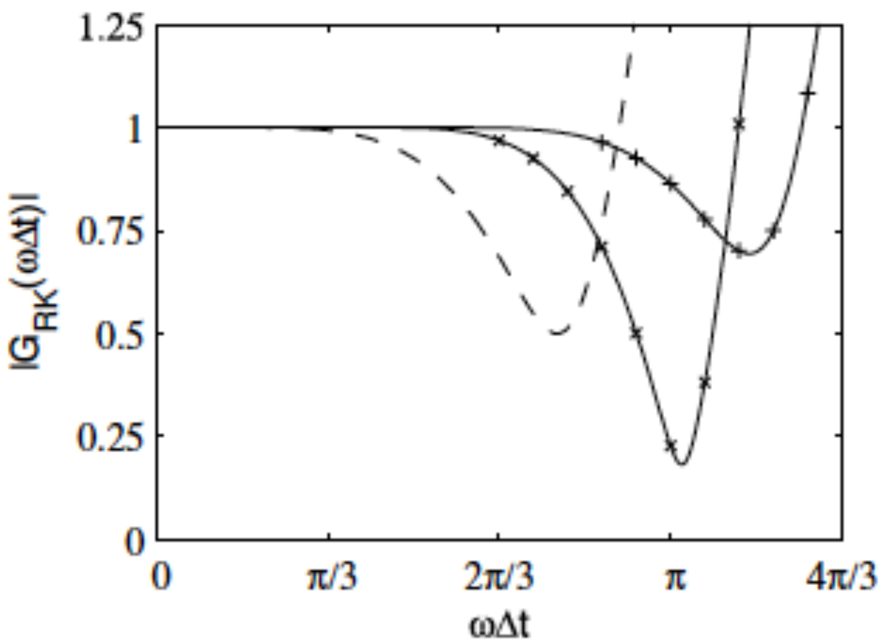
  Example openPSTD (www.openpstd.org)



4. Errors and stability

- **Numerical stability**, acoustic variables grow (unbounded) in time
- Criteria for stability in numerical methods from time iteration method with analysis per frequency

$$|G| = \left|\frac{S_j^{\tau}(\omega)}{S_j^{\tau-1}(\omega)}\right| \leq 1$$



[14]

4. Errors and stability

- **Numerical stability**,
- Courant number relates spatial to grid step (for equidistant grid spacing)

$$C \leq \frac{\Delta x}{\Delta t c_0}$$

- RK scheme (optimized 6-stage [14])

$$\Delta t \leq \frac{1}{0.8 f_{max}}$$

With

f_{max} = *the maximum frequeny supported by the spatial discretization method, e.g. in PS method*

$$f_{max} = \sqrt{D}\left(\frac{c_0}{\Delta x}\right)$$

With

D = *dimensionality of the problem*

$$\Rightarrow \quad C \leq \frac{1}{0.8\sqrt{D}}$$

4. Errors and stability

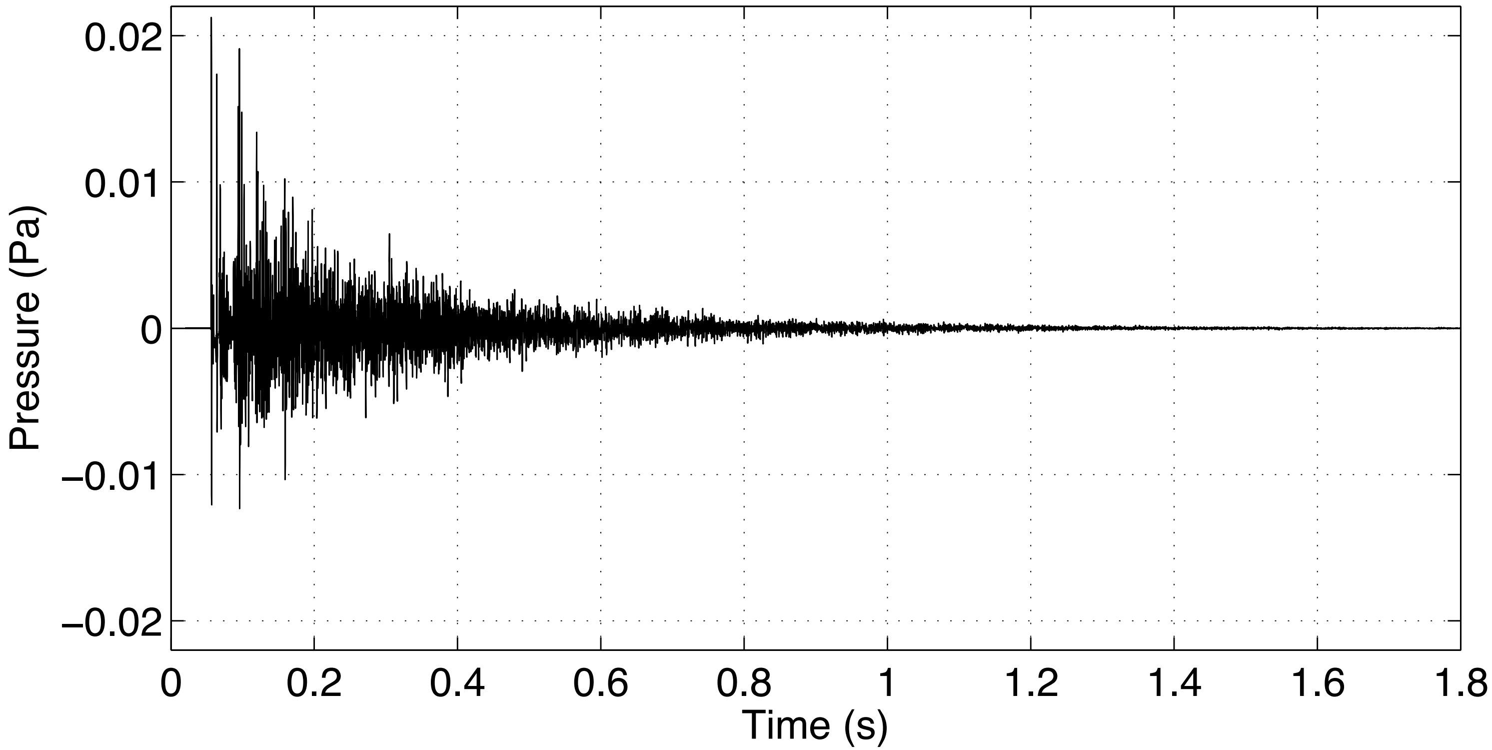
- **Numerical stability**,
- Leap Frog scheme in FDTD [11]

$$\Delta t \leq \frac{\Delta x}{\sqrt{D} c_0} \quad \Rightarrow \quad C \leq \frac{1}{\sqrt{D}}$$

- Aspects that influence stability:
  - Moving medium in outdoor acoustics
  - Inhomogeneous temperature
  - Small numerical errors that grow in time

5. Impulse and frequency response

- Green's functions are often sought by time-domain calculations
  i.e. Time response of environment to Dirac delta exitation



Example of a room impulse response

5. Impulse and frequency response

- Often used excitations in time-domain [3]

  Source function (see slide 12) at one location in space

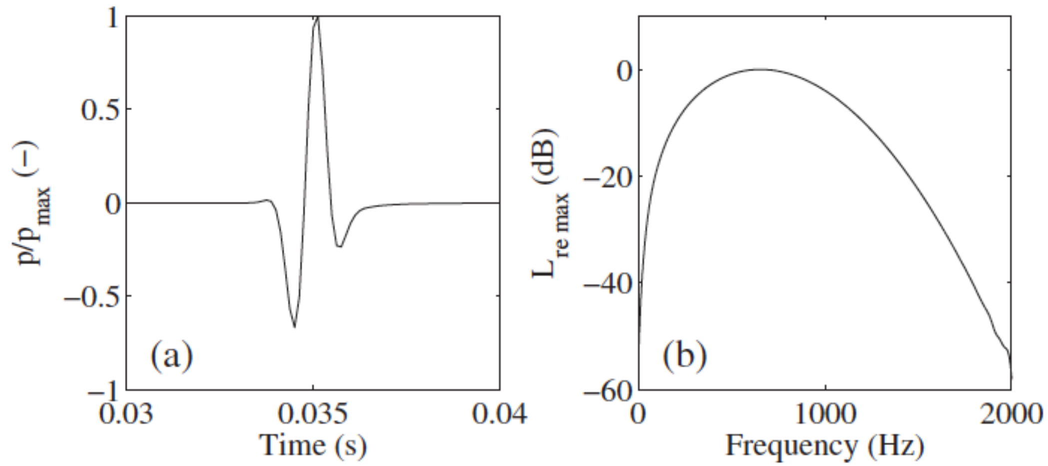
$$Q(t) = A\sin(\omega_c t)\, e^{-a(t-t_c)^2}$$

With

A = source amplitude

ω_c= angular center frequency

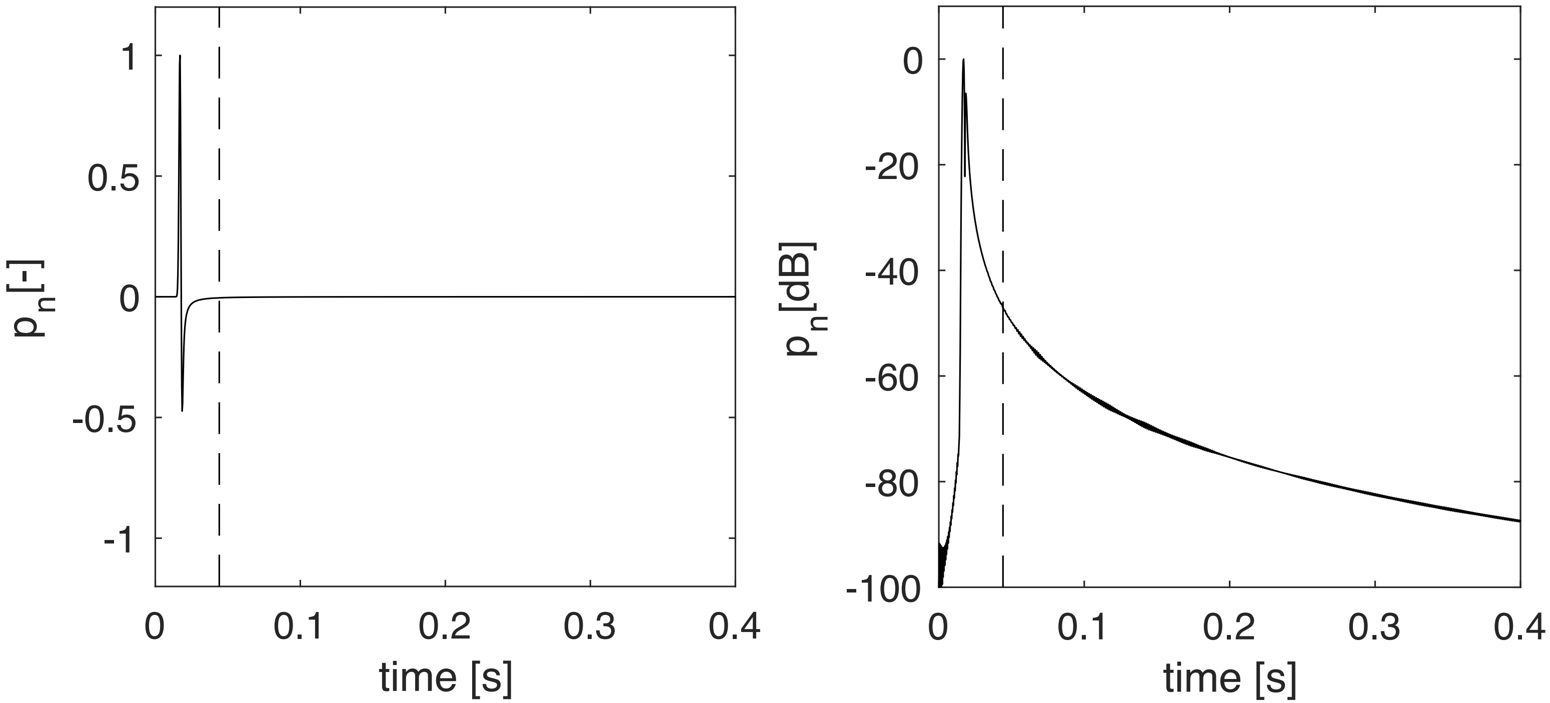
t_c = center time



Time and frequency response using source function in PSTD [3]

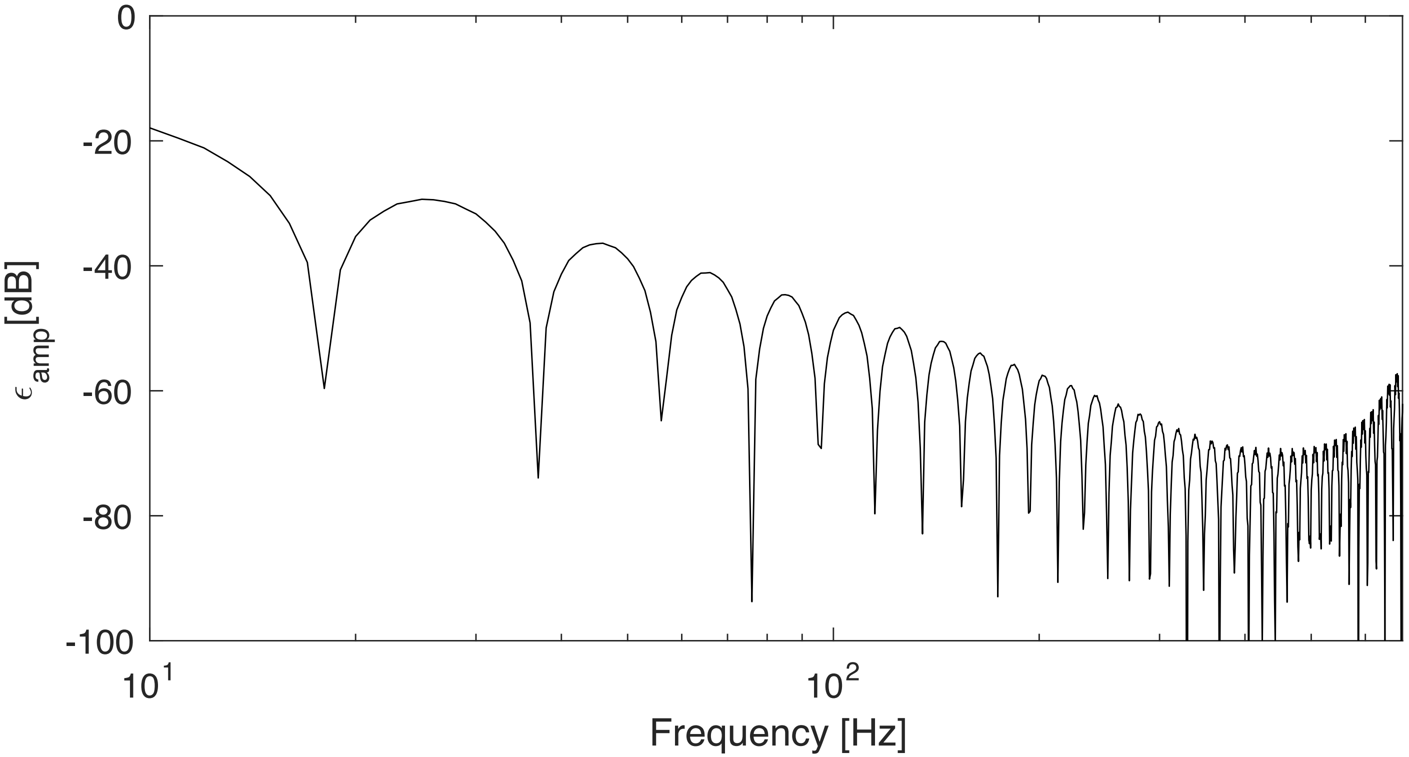
5. Impulse and frequency response

- 2D time free field response, from initial Gaussian pressure values (openPSTD calculation)
- The free field response only slowly decays!



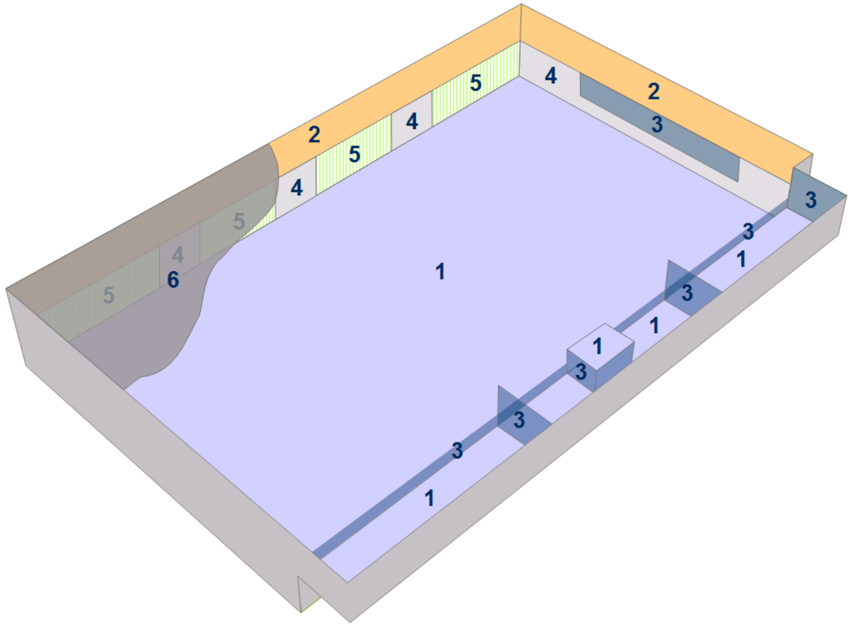
5. Impulse and frequency response

- 2D time free field response, from initial Gaussian pressure values (openPSTD calculation)
- Low frequency error!



6. Numerical examples

- Comparison: measurements in sportshall and PSTD method [37]

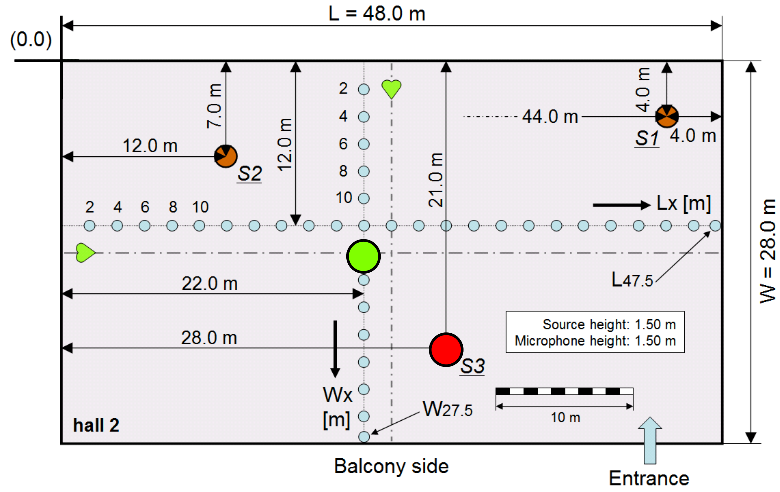


6. Numerical examples

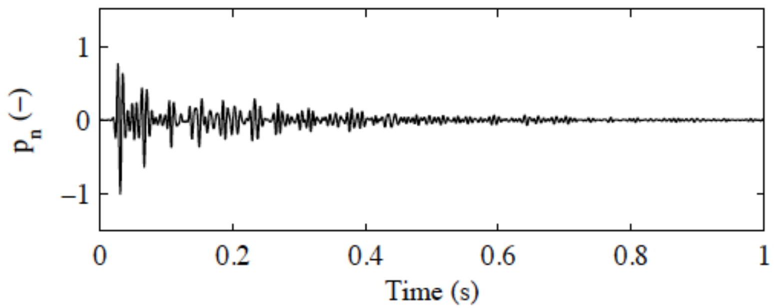
- Comparison: measurements in sportshall and PSTD method [37]

Receiver

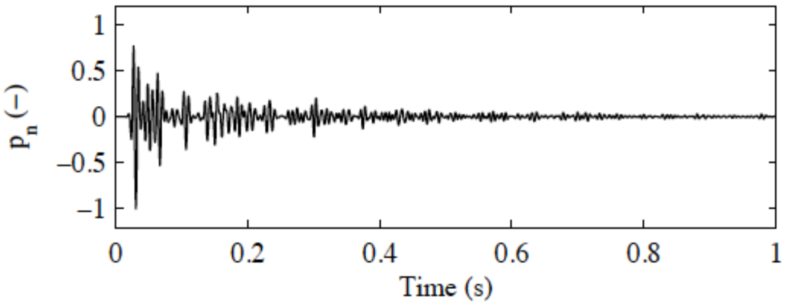
Source



PSTD



Measured



6. Numerical examples

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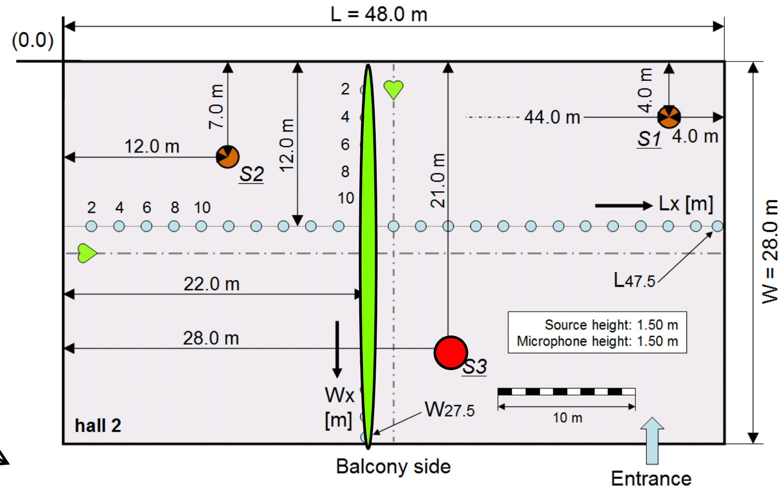
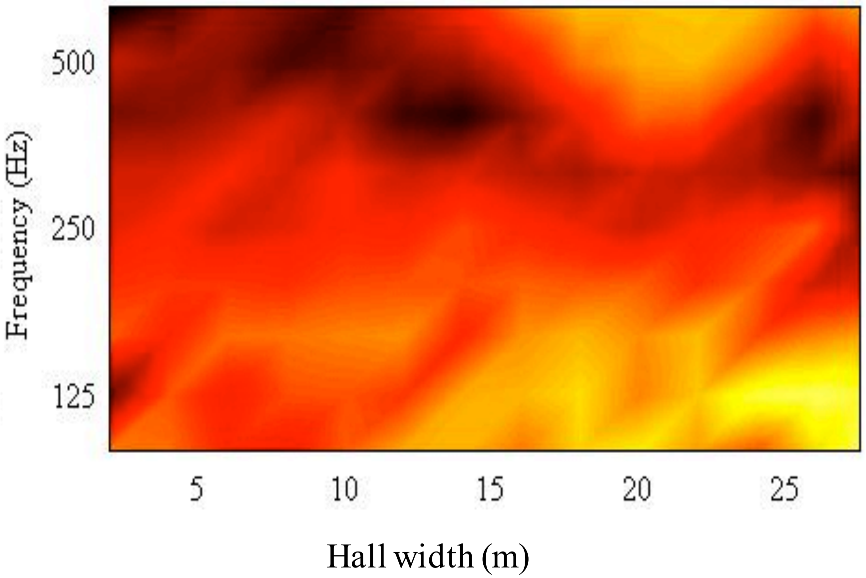
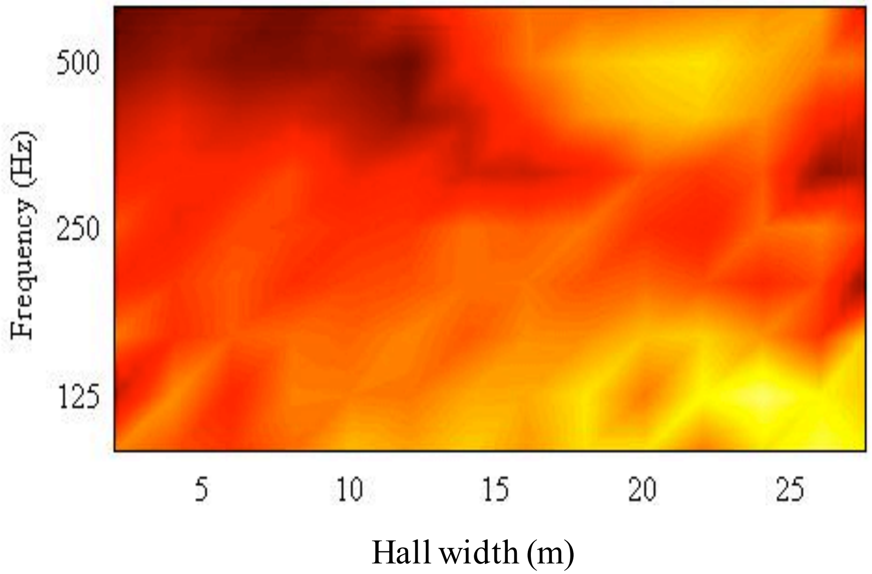
- Comparison: measurements in sportshall and PSTD method [37]



Measured

PSTD





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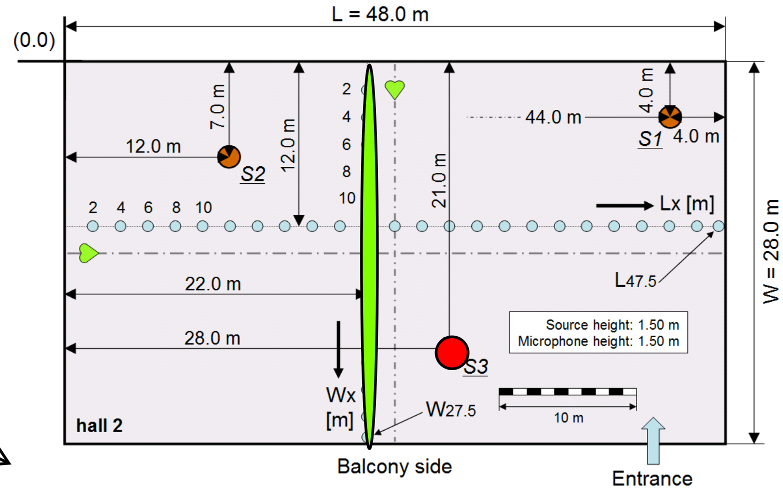
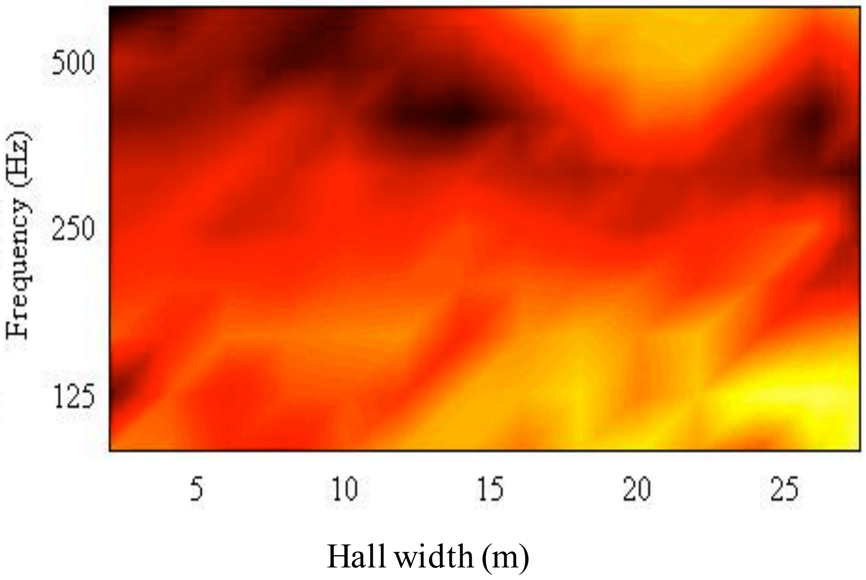
6. Numerical examples

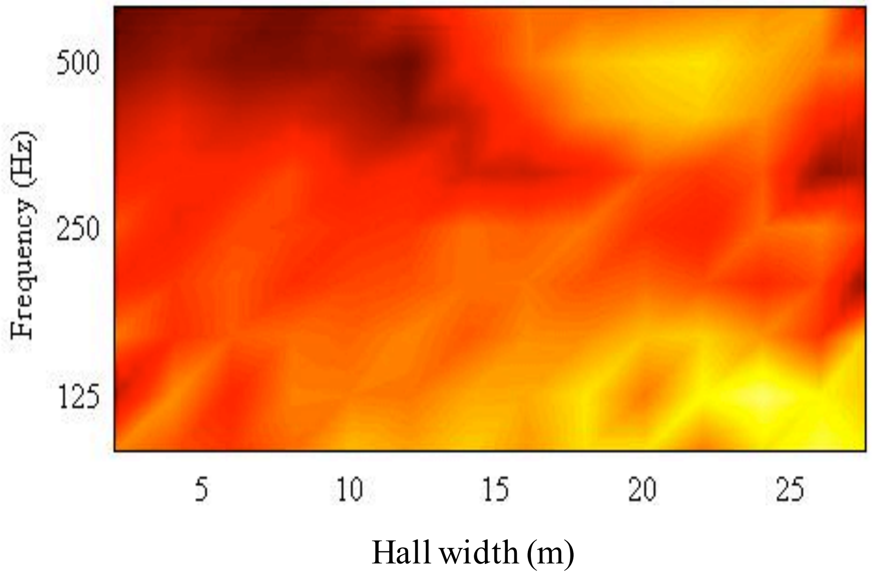
- Comparison: measurements in sportshall and PSTD method [37]



Measured

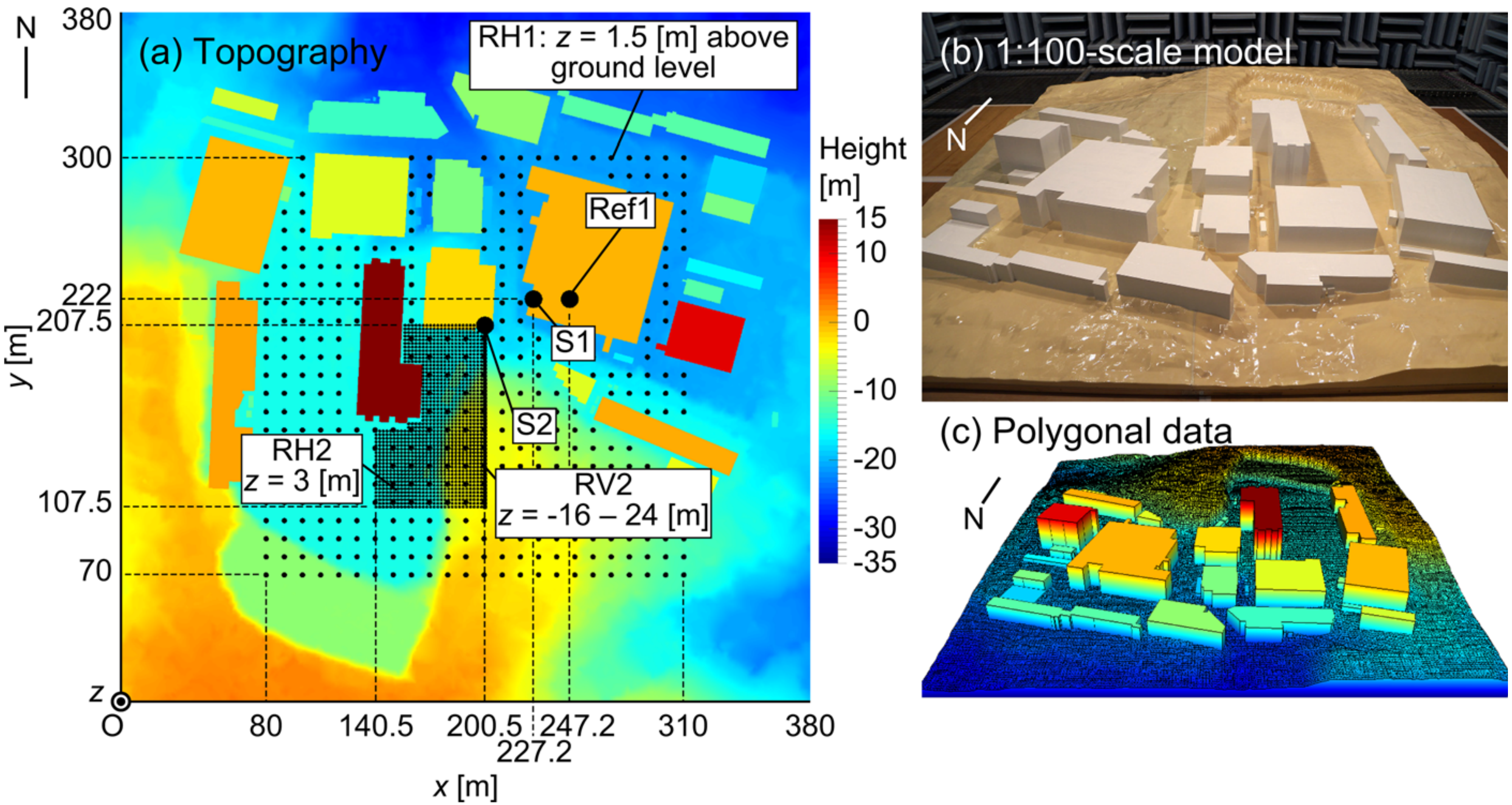
PSTD





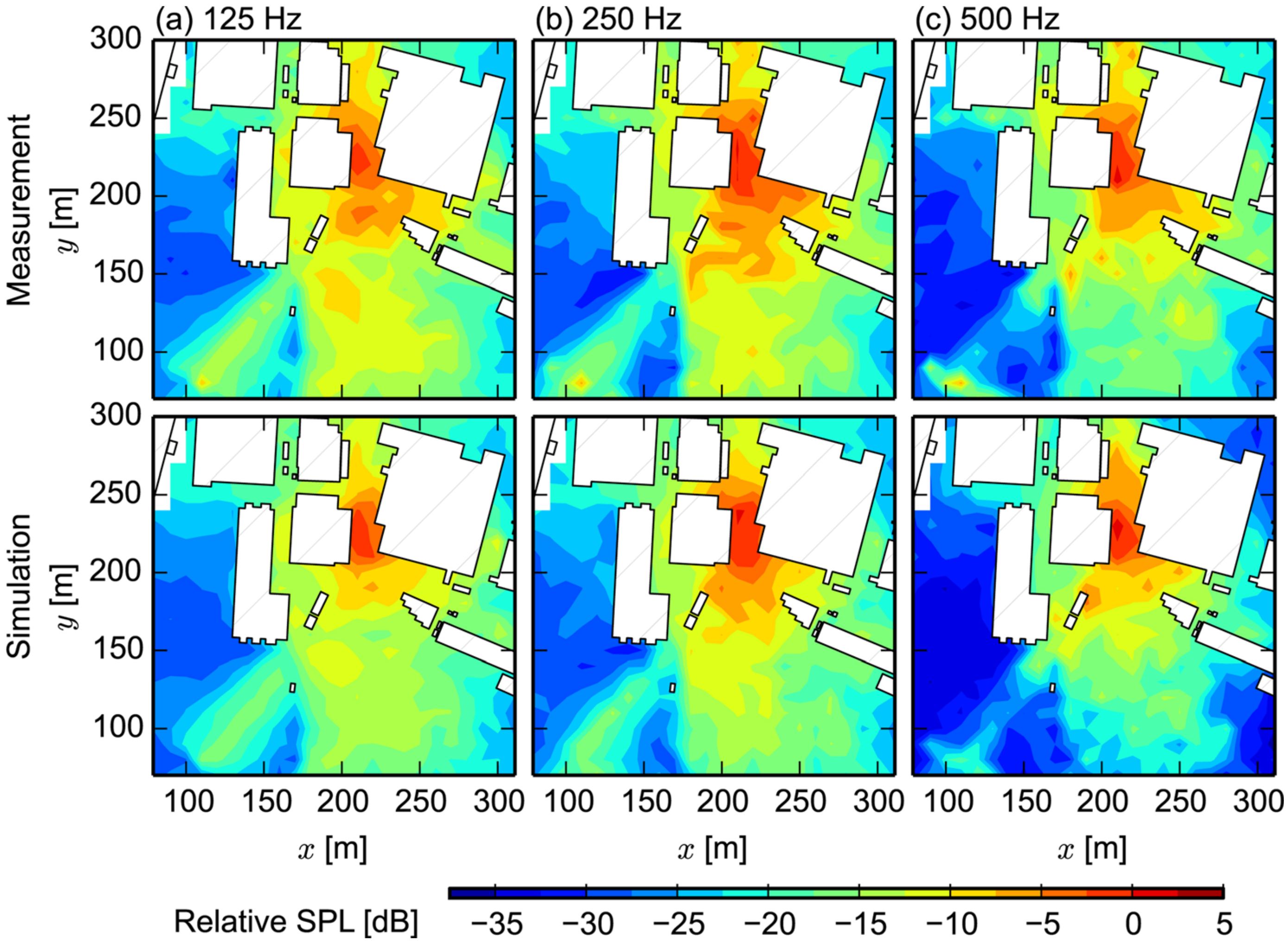
6. Numerical examples

- Comparison: scale model measurements urban area and FDTD method [38]



6. Numerical examples

- Comparison: scale model measurements urban area and FDTD method [38]



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